Handout 1
MATH 251

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1.1. Rates of Change

1.1.1. Performance Criteria

(a) Determine the average rates of change of a function algebraically from its expression.
(b) Determine the average rate of change of a function from its graph.
(c) Calculate average velocity from a set of data points over an interval.
(d) Calculate an estimate for the instantaneous velocity at a point from a set of data points.
(e) Approximate the instantaneous rate of change(or slope of the graph, or derivative) of a function at a point, from its graph.

1.1.2. Average rate of change

Example 1.1. Suppose that while climbing Mt. Shasta you note the temperature at the bottom and at the top of the mountain. Mt. Shasta is roughly 14,000 feet above sea level. At the base of the mountain (5,000 feet above sea level) the temperature is 45°F, while at the summit, the temperature is −15°F. What is the **average rate of change** of temperature while climbing?

Solution. It's easy to see that the temperature is a function of the height (the higher you go, the colder it gets). So, we first want to find out the change in temperature between the two points (base(t_b) and summit(t_s)). The change is

$$t_s - t_b = -15°\text{F} - 45°\text{F} = -60°\text{F}.$$

However, you are asked to find the **average** rate of change of temperature while you are climbing. The temperature changes due to change in height which is

$$h_s - h_b = 14,000 \text{ ft.} - 5,000 \text{ ft.} = 9,000 \text{ ft.}$$

Therefore, the average rate of change of temperature is the change in temperature with respect to the height i.e.

$$\textbf{average rate of change} = \frac{t_s - t_b}{h_s - h_b} = \frac{-60°\text{F}}{9,000 \text{ ft.}} = -0.0067°\text{F/foot}$$

□

In the previous example the temperature(t) was a function of the height(h). Therefore, t was the **dependent** variable and h was the **independent** variable. In general,

$$\textbf{Average rate of change} = \frac{\textbf{Change in the dependent variable}}{\textbf{Change in the independent variable}}$$

1.1.3. Average rate of change of a function

Consider a function $y = f(x)$. Recall that x is the independent variable and y is the dependent variable. Also consider two points x_1 and x_2 in the domain of f with $x_2 > x_1$. Let $y_1 = f(x_1)$ and $y_2 = f(x_2)$.

Then going by the previous definition of the average rate of change we have,

Definition 1.2. For a function $f(x)$ with points x_1, x_2 $(x_2 > x_1)$ in the domain of f, the **average rate of change of f with respect to x** over the interval $[x_1, x_2]$ is

$$\frac{\Delta f}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

Here Δf and Δx denote the change in f and x respectively.

Example 1.3. Consider the function $f(x) = -2x^2 + 4x - 1$. Find the average rate of change of f with respect to x from $x = 0$ to $x = 1$.

Solution. Here $x_1 = 0$ and $x_2 = 1$. Therefore,

$$\frac{\Delta f}{\Delta x} = \frac{f(1) - f(0)}{1 - 0} = \frac{1 - (-1)}{1} = 2$$

□

1.1.4. Average velocity

Consider a distance function $s(t)$ which denotes the distance of a particle from the starting point at time t. Between the time interval $[t_1, t_2]$, the average rate of change of the function $s(t)$ with respect to t is defined to be the **average velocity** of the particle, i.e.

Definition 1.4. The **average velocity** of a particle moving according to the distance function $s(t)$ between the time interval $[t_1, t_2]$ is

$$\frac{\Delta s}{\Delta t} = \frac{s(t_2) - s(t_1)}{t_2 - t_1}$$

Example 1.5. A ball dropped from a state of rest at time $t = 0$ travels a distance $s(t) = 4.9t^2$ m in t seconds. Compute the average velocity over the time interval $[3, 3.5]$.

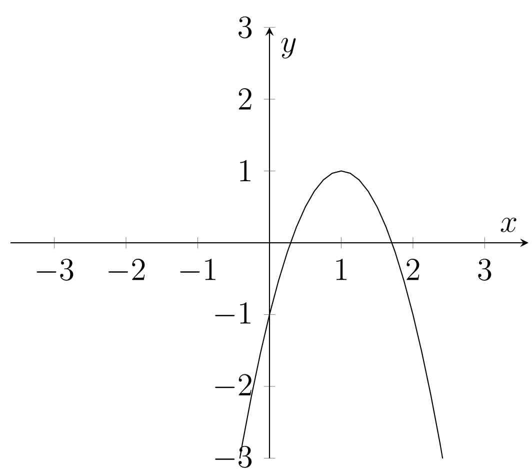
Solution. Here $t_1 = 3$ and $t_2 = 3.5$. The average velocity between these two points is therefore,

$$\frac{\Delta s}{\Delta t} = \frac{s(3.5) - s(3)}{3.5 - 3} = \frac{60.025 - 44.1}{0.5} = 31.85 \text{ m/s.}$$

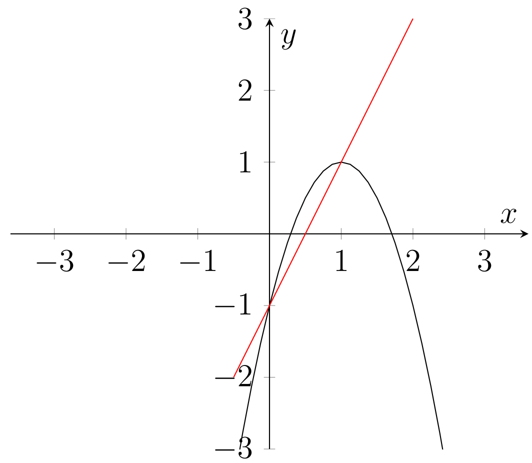
□

1.1.5. Secant Lines

Let's draw the graph of the function in Example 1.3.



If we plot the two points $(0, -1)$ and $(1, 1)$ on the graph and join them by a straight line, we end up having



This red line joining these two points on the graph of the function is called a **secant line**, and the slope of the secant line is the average rate of change of the function between those two points. (Recall how you found the slope of a line joining two points)

Hence, slope of the secant line is 2.

1.1.6. Instantaneous rate of change

Instantaneous rate of change is similar to the average rate of change that we looked at earlier *but* over **"very small"** intervals.

This is a very vague way to describe instantaneous rate of change, therefore, let's look at the earlier Example 1.3

Example 1.6. Consider the function $f(x) = -2x^2 + 4x - 1$. Find the instantaneous rate of change of f at $x = 0$.

Solution. Here we have just one point. As we mentioned in our "vague" definition, let's consider the small interval $[0, 0.1]$, and let's compute the average rate of change over this interval.

$$\left.\frac{\Delta f}{\Delta x}\right|_{[0,0.1]} = \frac{-0.62 - (-1)}{0.1 - 0} = \frac{0.38}{0.1} = 3.8$$

Now let's look at an even smaller interval $[0, 0.01]$. The average rate of change over this interval is,

$$\left.\frac{\Delta f}{\Delta x}\right|_{[0,0.01]} = \frac{-0.9602 - (-1)}{0.01 - 0} = \frac{0.0398}{0.01} = 3.98$$

We can keep on doing this for few more intervals each smaller than the previous. We construct a table involving these intervals.

Interval	$\frac{\Delta f}{\Delta x}$
$[0, 0.1]$	3.8
$[0, 0.01]$	3.98
$[0, 0.001]$	3.998
$[0, 0.0001]$	3.9998
$[0, 0.00001]$	3.99998

The above table clearly shows that, as the intervals get smaller and smaller, the difference quotient $\frac{\Delta f}{\Delta x}$ gets closer to 4. This suggests that 4 is a good candidate for the instantaneous rate of change of the function f at $x = 0$.

□

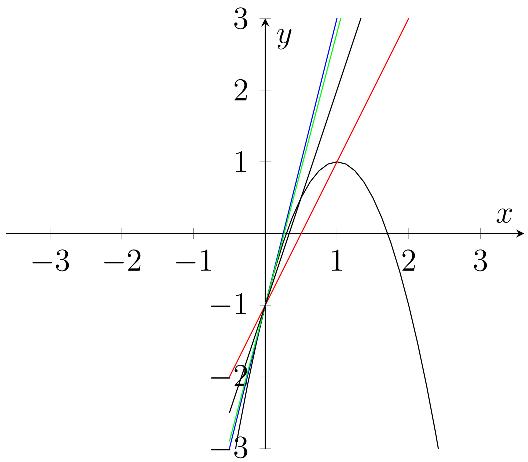
We can now define the instantaneous rate of change in a slightly better way

Definition 1.7. The **instantaneous rate of change** is the *limit* (a concept we will be learning soon) of the average rate of change as the length of the intervals shrink to zero.

Similarly the **instantaneous velocity** is the *limit* of the average velocity as the length of the time intervals shrink to zero. The instantaneous velocity also describes the velocity of a particle at that instant of time.

1.1.7. Tangent Lines

If we go back to the graph of our function from Example 1.3, and keep drawing secant lines for every small interval, we end up having



If we look closely at the graph, then we can see that the red secant line "approaches" the blue tangent line at the point $x = 0$. This happens as the length of the interval between the two points (where the left end point is fixed at 0 and the right end point approaches 0) gets smaller and smaller.

This can be verified by looking at the black and green secant lines which are closer to the blue tangent line than the original red secant line.

So what about the slope of this tangent line at $x = 0$? If we recall, we previously concluded that the slope of the secant line joining two points is precisely the average rate of change of the function between those two points. We also defined the instantaneous rate of change to be *limit* of the average rate of change over very small intervals, which we computed in the second column of the table above. Thus,

Remark. The slope of the tangent line of a function $f(x)$ at $x = a$ is the instantaneous rate of change of the function at $x = a$.

1.2. Limits

1.2.1. Performance Criteria

(a) Use the graph of a function to determine left and right hand limits at a point.
(b) Use left and right hand limits obtained from the graph of a function to determine a limit at a point.
(c) Use limit notation correctly.
(d) Calculate left and right hand limits.
(e) Calculate limits at a point.
(f) Calculate limits at infinity.

1.2.2. Definition of a Limit

In the previous section, it was mentioned that the instantaneous rate of change is the *limit* of the average rate of change. We formally define the concept of *limit* in this section.

Consider the function $f(x) = \dfrac{e^x - 1}{x}$. Notice that $f(0)$ is not defined. When we set $x = 0$ we have the undefined expression $0/0$. However, let's try to find out what happens when x *approaches* 0.

1.2.2.1. **Left hand limit.** We first choose values of x which are close to 0 but less than 0. In this case, we say we are approaching 0 **from the left**. Let's see the values of the function values corresponding to these values of x.

x	$\frac{e^x - 1}{x}$
-0.1	0.951625
-0.01	0.995016
-0.001	0.9995
-0.0001	0.99995
-0.00001	0.999995

It is clear from the above table that as x gets closer and closer to 0 from the left, the value of $\frac{e^x - 1}{x}$ gets closer to 1. We say that the **left hand limit** of the function f is 1. We write it as,

$$\lim_{x \to 0^-} \frac{e^x - 1}{x} = 1$$

Here lim is an abbreviation for limit and $x \to 0^-$ means x approaches 0 from the left.

1.2.2.2. **Right hand limit.** Now let's choose values of x which are close to 0 but greater than 0. In this case, we say we are approaching 0 **from the right**. Let's see the function values corresponding to these values of x.

x	$\frac{e^x - 1}{x}$
0.1	1.051709
0.01	1.00501
0.001	1.0005
0.0001	1.00005
0.00001	1.000005

Again, it is clear from the above table that as x gets closer and closer to 0 from the right, the value of $\frac{e^x - 1}{x}$ gets closer to 1. We say that the **right hand limit** of the function f is 1. We write it as,

$$\lim_{x \to 0^+} \frac{e^x - 1}{x} = 1$$

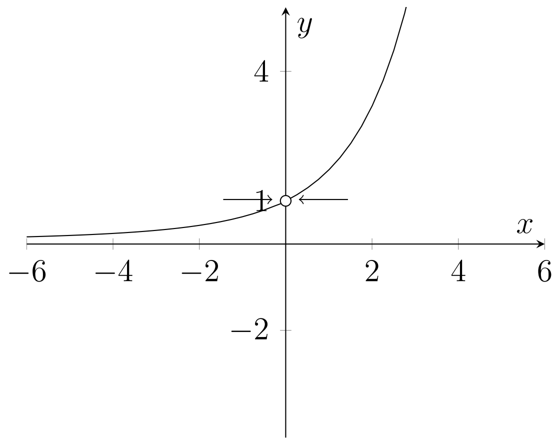
Here $x \to 0^+$ means x approaches 0 from the right.

Since both left hand limit and right hand limit of the above function give the same finite value (which is 1), we say that **limit** of the function **exists** and is equal to that finite value. We write it as,

$$\lim_{x \to 0} \frac{e^x - 1}{x} = 1$$

Note that we don't use $+$ or $-$ with 0 anymore since we are denoting the limit of the function.

Additionally, if we look at the graph of the function $f(x) = \dfrac{e^x - 1}{x}$, we see that as we approach 0 from either side the graph approaches 1 (without being defined at $x = 0$).



Let's formally define limit for an arbitrary function $f(x)$.

Definition 1.8. Assume that $f(x)$ is defined for all x in an open interval containing a, but not necessarily at a itself. We say that

the limit of f(x) as x approaches a is equal to L

if $\mid f(x) - L \mid$ becomes arbitrarily small when x is any number sufficiently close (bit not equal) to a. In this case, we write

$$\lim_{x \to a} f(x) = L$$

If the values of $f(x)$ do not converge to any limit as $x \to a$, we say that $\lim_{x \to a} f(x)$ **does not exist**.

Remark. For a limit $\lim_{x \to a} f(x) = L$ to exist, it is necessary that both the left hand limit $\lim_{x \to a^-} f(x)$ and right hand limit $\lim_{x \to a^+} f(x)$ exist and is equal to L.

Example 1.9. Consider the function $f(x) = \dfrac{x}{\mid x \mid}$. Does $\lim_{x \to 0} f(x)$ exist?

Solution. By the definition of the absolute value, for $x < 0, \mid x \mid = -x$. Therefore,

$$\lim_{x \to 0^-} \frac{x}{\mid x \mid} = \lim_{x \to 0^-} \frac{x}{-x} = \lim_{x \to 0^-} -1 = -1.$$

Recall, that $x \to 0^-$ means we are approaching 0 from the left i.e. with numbers less that 0, hence the reason for $\mid x \mid = -x$.

On the other hand, for $x > 0, \mid x \mid = x$. Therefore,

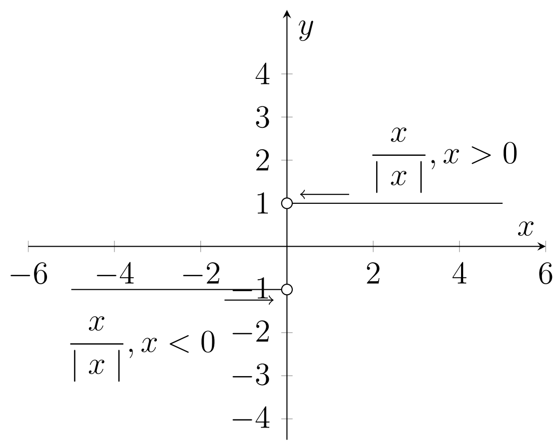
$$\lim_{x\to 0^+} \frac{x}{\mid x \mid} = \lim_{x\to 0^+} \frac{x}{x} = \lim_{x\to 0^+} 1 = 1.$$

Since,

$$\lim_{x\to 0^-} \frac{x}{\mid x \mid} \neq \lim_{x\to 0^+} \frac{x}{\mid x \mid}$$

therefore, $\lim_{x\to 0} \dfrac{x}{\mid x \mid}$ doesn't exist.

Additionally, looking at the graph of $f(x) = \dfrac{x}{\mid x \mid}$ below verifies the same result.



□

1.2.3. Infinite Limits

Sometimes functions $f(x)$ tend to ∞ or $-\infty$ as x approaches a value a. In this case, we say that $\lim_{x\to a} f(x)$ does not exist. More precisely,

- $\lim_{x\to a} f(x) = \infty$, if $f(x)$ increases without bound as $x \to a$.
- $\lim_{x\to a} f(x) = -\infty$, if $f(x)$ decreases without bound as $x \to a$.

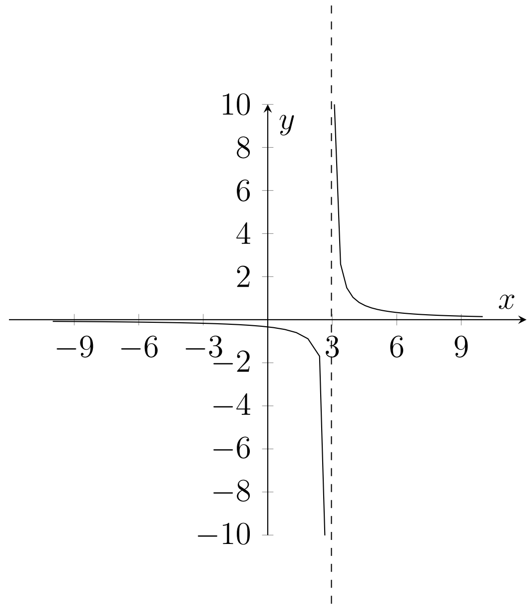
Even though the limit does not exist, we still say that $f(x)$ has an *infinite limit.*

When $f(x)$ approaches ∞ or $-\infty$ from one or both sides, the line $x = a$ (which is vertical) is called a **vertical asymptote**

Example 1.10. Investigate the one-sided limit graphically,

$$\lim_{x\to 3^+} \frac{1}{x-3}$$

Solution. Let us draw the graph of $f(x) = \dfrac{1}{x-3}$.



As we move closer to 3 from the right we see that the graph shoots up to ∞. Therefore,

$$\lim_{x\to 3^+} \frac{1}{x-3} = \infty$$

The line $x = 3$ is a vertical asymptote.

$\square$

Additional Problem. Can you find the other one-sided limit?

$$\lim_{x\to 3^-} \frac{1}{x-3}$$

1.2.4. Limit Laws

There are some basic limit laws that you would like to follow when evaluating limits.

If $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist, then

- **Sum Law:** $\lim_{x\to a}(f(x)+g(x))$ exists and

$$\lim_{x\to a}(f(x)+g(x)) = \lim_{x\to a} f(x) + \lim_{x\to} g(x)$$

- **Product Law:** $\lim_{x\to a} f(x)g(x)$ exists and

$$\lim_{x\to a} f(x)g(x) = (\lim_{x\to a} f(x))(\lim_{x\to a} g(x))$$

- **Quotient Law:** If $\lim_{x\to a} g(x) \neq 0$, then $\lim_{x\to a} \frac{f(x)}{g(x)}$ exists and

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$$

- **Powers and Roots:** If p, q are integers with $q \neq 0$, then $\lim_{x\to a} [f(x)]^{p/q}$ exists and

$$\lim_{x\to a} [f(x)]^{p/q} = (\lim_{x\to a} f(x))^{p/q}$$

Example 1.11. Evaluate

$$\lim_{x\to 2} \frac{x^2 + 3x - 9}{x - 4}$$

Solution. Let's use the limit laws here,

$$\lim_{x\to 2} \frac{x^2 + 3x - 9}{x - 4} = \frac{\lim_{x\to 2}(x^2 + 3x - 9)}{\lim_{x\to 2}(x - 4)} = \frac{4 + 6 - 9}{2 - 4} = -\frac{1}{2}$$

□

Usually the first thing to do when you are doing a problem in limit is to substitute the value of x into the expression and see what the result is. If you end up getting a finite value, then stop right there and that is the limit. However, if you have an undefined case like $0/0$, then you have to explore other methods.

Handout 3
MATH 251

Dibyajyoti Deb

3.1. Trigonometric Limits

3.1.1. Performance Criteria

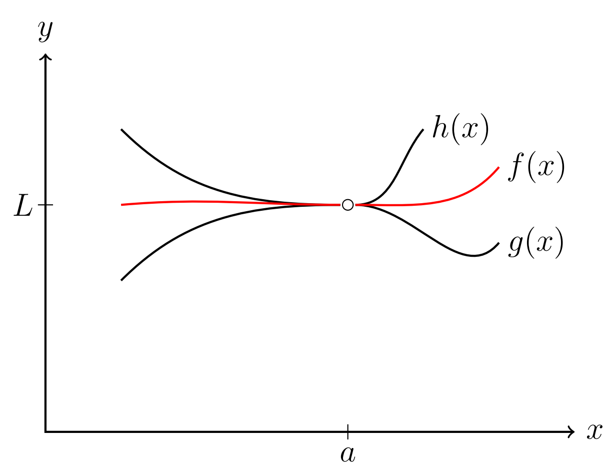
(a) Determine limits using the Squeeze Theorem.
(b) Calculate limits involving trigonometric functions.

3.1.2. The Squeeze Theorem

Let us say that we have a function $f(x)$ that is "squeezed" between two functions $g(x)$ and $h(x)$ on an interval I, i.e.,

$$g(x) \leq f(x) \leq h(x) \quad \text{for all } x \in I \text{ and}$$

$$\lim_{x\to a} g(x) = \lim_{x\to a} h(x) = L$$



It is easy to see from the graph above that the function $f(x)$ is "squeezed" between $h(x)$ and $g(x)$. If this happens then we have the Squeeze Theorem,

Theorem 3.1. *Assume that for $x \neq a$ (in some open interval containing a),*

$$g(x) \leq f(x) \leq h(x) \quad \text{and} \quad \lim_{x\to a} g(x) = \lim_{x\to a} h(x) = L$$

Then $\lim_{x\to a} f(x)$ exists and $\lim_{x\to a} f(x) = L$.

It should be somewhat clear from the previous diagram as to why this theorem is true (The diagram doesn't give us a formal proof though).

Example 3.2. Evaluate using the Squeeze Theorem.

$$\lim_{x\to 0} x \sin \frac{1}{x^2}$$

Solution. The goal here is to find two functions $g(x)$ and $h(x)$ such that

$$g(x) \leq x \sin \frac{1}{x^2} \leq h(x) \quad \text{and} \quad \lim_{x\to 0} g(x) = \lim_{x\to 0} h(x) = L$$

We know that $-1 \leq \sin \frac{1}{x^2} \leq 1$ for all $x \neq 0$. Therefore, $|\sin \frac{1}{x^2}| \leq 1$ for all $x \neq 0$. If we multiply by $|x|$, we have $|x \sin \frac{1}{x^2}| \leq |x|$ and therefore,

$$-|x| \leq x \sin \frac{1}{x^2} \leq |x|$$

Now $\lim_{x\to 0} |x| = 0$ and $\lim_{x\to 0}(-|x|) = 0$, therefore we can apply Squeeze Theorem to conclude that $\lim_{x\to 0} x \sin \frac{1}{x^2} = 0$.

□

3.1.3. Important Trigonometric Limits

There are two important trigonometric limits that we will be using throughout this section and beyond. They are,

$$\lim_{x\to 0} \frac{\sin x}{x} = 1, \qquad \lim_{x\to 0} \frac{1 - \cos x}{x} = 0$$

The proof of the above results use the Squeeze Theorem and the fact that

$$\cos x \leq \frac{\sin x}{x} \leq 1 \quad \text{for} \quad -\frac{\pi}{2} < x < \frac{\pi}{2}, \quad x \neq 0$$

Example 3.3. Evaluate the limit

$$\lim_{x\to 0} \frac{x^2}{\sin^2 x}$$

Solution. Initial substitution with $x = 0$ in the expression yields the indeterminate form 0/0. Hence we have to simplify the expression. The goal here is to use one of the two trigonometric limits that we discussed earlier. Note that both x and $\sin x$ is present in the expression.

$$\begin{aligned}
\lim_{x\to 0} \frac{x^2}{\sin^2 x} &= \lim_{x\to 0} \frac{1}{\frac{\sin^2 x}{x^2}} \\
&= \lim_{x\to 0} \frac{1}{\left(\frac{\sin x}{x}\right)^2} \\
&= \frac{\lim_{x\to 0} 1}{\lim_{x\to 0} \left(\frac{\sin x}{x}\right)^2} \\
&= \frac{1}{1^2} = 1.
\end{aligned}$$

□

Example 3.4. Evaluate the limit

$$\lim_{x\to 0} \frac{1-\cos x}{\sin x}$$

Solution. Initial substitution with $x = 0$ in the expression yields the indeterminate form 0/0. Hence we have to simplify the expression. The goal here is to use one of the two trigonometric limits that we discussed earlier. Note that both $1 - \cos x$ and $\sin x$ are present in the expression. Therefore we somehow want to get x in the denominator of each of these terms.

$$\begin{aligned}
\lim_{x\to 0}\frac{1-\cos x}{\sin x} &= \lim_{x\to 0}\frac{\frac{1-\cos x}{x}}{\frac{\sin x}{x}} \text{ (Dividing both the numerator and denominator by } x\text{)} \\
&= \frac{\lim_{x\to 0}\frac{1-\cos x}{x}}{\lim_{x\to 0}\frac{\sin x}{x}} \\
&= \frac{0}{1} = 0.
\end{aligned}$$

□

Example 3.5. Evaluate the limit

$$\lim_{x\to 0}\frac{\tan 4x}{9x}$$

Solution. Initial substitution with $x = 0$ in the expression yields the indeterminate form $0/0$. Hence we have to simplify the expression. Let us try to get everything in terms of $\sin x$ and $\cos x$.

$$\frac{\tan 4x}{9x} = \frac{\frac{\sin 4x}{\cos 4x}}{9x} = \frac{\sin 4x}{9x\cos 4x}$$

Now we know that $\lim_{x\to 0}\frac{\sin x}{x} = 0$. Since $x \to 0$, therefore, $4x \to 0$ as $4x$ is just a multiple of x. Therefore, if $4x = h$, then we can say that

$$\lim_{h\to 0}\frac{\sin h}{h} = 1 \text{ where } h = 4x.$$

We can now rewrite our initial problem as

$$\begin{aligned}
\lim_{x\to 0}\frac{\tan 4x}{9x} &= \lim_{x\to 0}\frac{\sin 4x}{9x\cos 4x} \\
&= \lim_{x\to 0}\left(\frac{\sin 4x}{4x}\right)\left(\frac{4x}{9x\cos x}\right) \\
&= \lim_{h\to 0}\left(\frac{\sin h}{h}\right)\lim_{x\to 0}\left(\frac{4x}{9x\cos x}\right) \\
&= 1\cdot\lim_{x\to 0}\left(\frac{4}{9\cos x}\right) \\
&= 1\cdot\frac{4}{9\cdot 1} = \frac{4}{9}
\end{aligned}$$

□

3.2. Limits at Infinity

3.2.1. Performance Criteria

(a) Calculate limits at infinity.
(b) Calculate horizontal asymptotes.

3.2.2. Limits at infinity

We have solved quite a few problems involving limits in the previous sections. In all these problems we considered limits where x approached a finite number a. Now, we consider limits where x approaches either ∞ or $-\infty$. Before we delve into a theorem involving limits at infinity, we look at some basic results involving simple functions.

Theorem 3.6. *For all* $n > 0$,

$$\lim_{x\to\infty} x^n = \infty, \qquad \lim_{x\to\pm\infty} x^{-n} = \lim_{x\to\pm\infty} \frac{1}{x^n} = 0$$

If n *is a whole number then,*

$$\lim_{x\to-\infty} x^n = \begin{cases} \infty, & \text{if } n \text{ is even} \\ -\infty, & \text{if } n \text{ is odd} \end{cases}$$

The best way to verify these results is by substituting a very large positive number for x when x approaches ∞ and substituting a negative number with a large absolute value when x approaches $-\infty$ and checking the answer.

Example 3.7. Find

$$\lim_{x\to-\infty} \frac{2}{x^3}$$

Solution. $\lim_{x\to-\infty} \frac{2}{x^3} = 2 \lim_{x\to-\infty} \frac{1}{x^3} = 2 \cdot 0 = 0$. (By Theorem 3.6) Intuitively you can think of x as -10^9 (a negative number with a very large absolute value). So, $1/x^3 = -1/10^{27}$ which is a very very small number hence very close to 0.

□

We will be mostly concerned with limits of rational functions. Rational functions are fractions in which both the numerator and denominator are polynomials in x. In this regard we have the following theorem.

Theorem 3.8. *The limit of a rational function as x approaches either ∞ or $-\infty$ depends only on the leading term of its numerator and denominator, i.e, if $a_n, b_m \neq 0$, then*

$$\lim_{x\to\pm\infty} \frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_0} = \frac{a_n}{b_m} \lim_{x\to\pm\infty} x^{n-m} = L$$

$$\textit{where } L = \begin{cases} 0, & \textit{if } n < m \\ \dfrac{a_n}{b_m}, & \textit{if } n = m \\ \infty \textit{ or } -\infty, & \textit{if } n > m \end{cases}$$

Let us look at some applications of the above theorem,

Example 3.9. Evaluate the limit

$$\lim_{x\to\infty} \frac{3x^{9/2} - 4x^3 + 2x^2 - 9}{4x^2 - 3x + 7}$$

Solution. Here the leading term of numerator is $3x^{9/2}$ and that of the denominator is $4x^2$. Since $9/2 > 4$, therefore the limit is either ∞ or $-\infty$. Again, by Theorem 3.8, we have

$$\lim_{x\to\infty} \frac{3x^{9/2} - 4x^3 + 2x^2 - 9}{4x^2 - 3x + 7} = \frac{3}{4} \lim_{x\to\infty} x^{9/2-2} = \frac{3}{4} \lim_{x\to\infty} x^{5/2} = \infty.$$

□

Example 3.10. Evaluate the limit

$$\lim_{x\to\infty} \frac{\sqrt{9x^4 + 3x^2 + 2}}{4x^3 - 1}$$

Solution. Here we can think of the leading term of the numerator as $\sqrt{9x^4} = 3x^2$ and the leading term of the denominator is $4x^3$. Since $2 < 3$, therefore

$$\lim_{x\to\infty} \frac{\sqrt{9x^4 + 3x^2 + 2}}{4x^3 - 1} = \frac{3}{4} \lim_{x\to\infty} x^{-1} = \frac{3}{4} \cdot 0 = 0.$$

□

3.2.3. Horizontal asymptote

Horizontal asymptotes are horizontal straight lines that are parallel to the x-axis which are never touched by any graph except at ∞ or $-\infty$, i.e, the graph gets closer and closer to the horizontal asymptote as x approaches ∞ or $-\infty$. So, how do we find the horizontal asymptotes?

Definition 3.11. A horizontal line $y = L$ is a **horizontal asymptote** if

$$\lim_{x\to\infty} f(x) = L \quad \text{and/or} \quad \lim_{x\to-\infty} f(x) = L$$

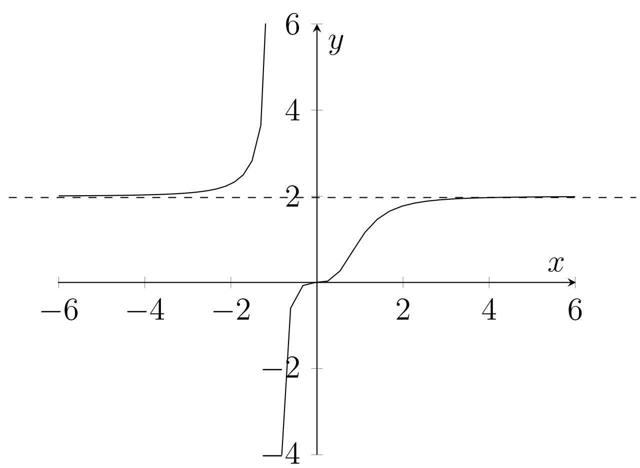
Example 3.12. Find the horizontal asymptote(s).

$$f(x) = \frac{2x^3}{x^3+1}$$

Solution. By Theorem 3.8 we have,

$$\lim_{x\to\pm\infty} \frac{2x^3}{x^3+1} = 2 \lim_{x\to\pm\infty} \frac{x^3}{x^3} = 2 \cdot 1 = 2.$$

Therefore, $y = 2$ is a horizontal asymptote of the function $f(x) = \dfrac{2x^3}{x^3+1}$. The graph of the function looks like,



As can be seen from the graph above that the line $y = 2$ is a horizontal asymptote.

□

Handout 2
MATH 251

Dibyajyoti Deb

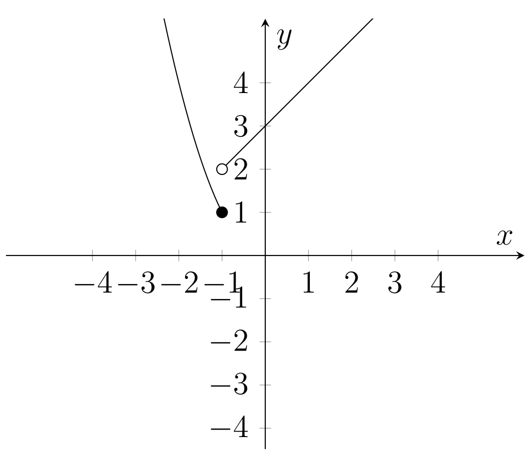
2.1. Limits and Continuity

2.1.1. Performance Criteria

(a) Determine and classify points of discontinuity from the graph of a function.
(b) Calculate points of discontinuity.
(c) Prove one-sided continuity.
(d) Show continuity at a point.

2.1.2. Continuity

A function $f(x)$ is said to be **continuous** everywhere, if the graph of the function does not have a "break" in it. What do I mean by not having a "break"? Look at the graph below



The graph has a break at the point -1. We say that the graph has a **discontinuity** at $x = -1$.

Discontinuity at a point on a graph occurs because the left hand and right hand limit of $f(x)$ as x approaches a are not equal and hence $\lim_{x\to -1} f(x)$ does not exist. In the above example, $\lim_{x\to -1^-} f(x) = 1$ whereas $\lim_{x\to -1^+} f(x) = 2$, hence the limit $\lim_{x\to -1} f(x)$ does not exist. On the other hand,

Definition 2.1. Assume that $f(x)$ is defined on an open interval containing the point a. Then f is **continuous** at $x = a$ if

$$\lim_{x\to a} f(x) = f(a)$$

If the limit does not exist, or if it exists but is not equal to $f(a)$, we say that f has a **discontinuity**(or is **discontinuous**) at $x = a$.

Therefore, to check continuity of a function at a point $x = a$, we need to check three conditions.

- $f(a)$ is defined.
- $\lim_{x\to a} f(x)$ exists.
- They are equal to each other i.e. $\lim_{x\to a} f(x) = f(a)$.

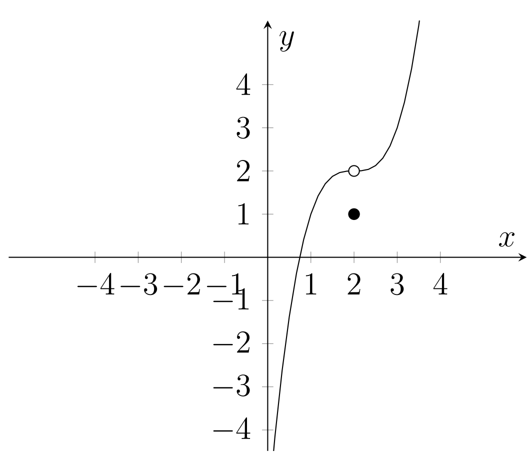
2.1.3. Types of discontinuities

In general, there are usually three types of discontinuity that can be associated with a function.

- **Removable discontinuity** - This happens when the limit of the function at a point a is not equal to $f(a)$, i.e.

$$\lim_{x\to a} f(x) \neq f(a)$$

Example of a graph of such a function would be



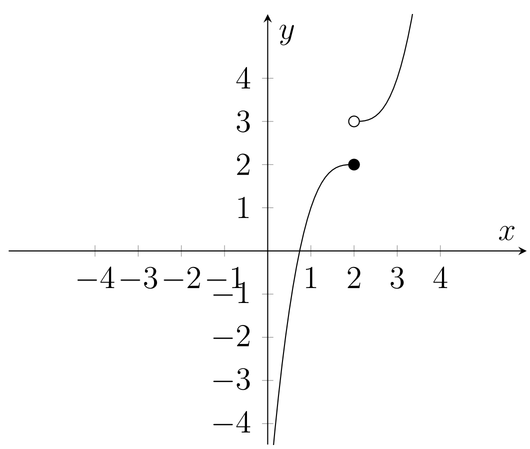
As is clear from the above graph, the function is not continuous at $x = 2$. This is because even though $\lim_{x\to 2} f(x) = 2$, it is not equal to $f(2)$ which is 1.

The reason this type of discontinuity is called "removable" is because the discontinuity can be removed by defining the value of the function at that point to be equal to the value of the limit. Therefore in the above graph, we can define $f(2) = 2$ and then it would equal $\lim_{x\to 2}$ which would remove the discontinuity at $x = 2$.

- **Jump discontinuity** - This happens when the one-sided limits exist but are not equal to each other, i.e.

$$\lim_{x\to a^-} f(x) \neq \lim_{x\to a^+} f(x)$$

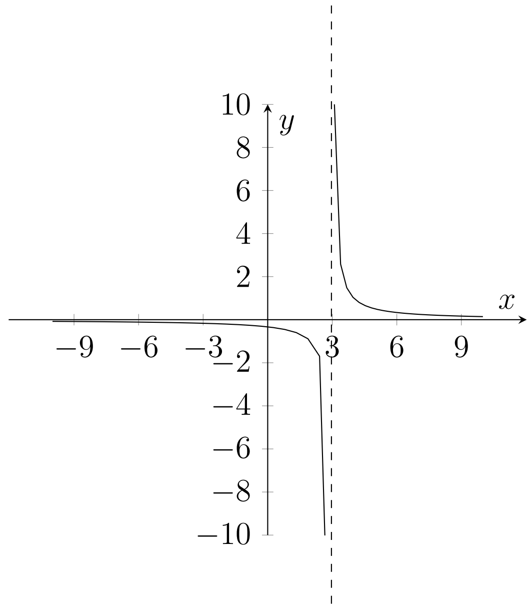
Example of a graph of such a function would be



It's clear from the above graph that $\lim_{x\to 2^-} f(x) = 2$ but $\lim_{x\to 2^+} f(x) = 3$, hence $\lim_{x\to 2} f(x)$ does not exist. Therefore $f(x)$ cannot be continuous at $x = 2$. Discontinuities of this type cannot be removed unlike the previous case.

- **Infinite discontinuity** - This happens when one or both of the one sided limits is infinite at $x = a$.

  Example of a graph of such a function would be



It's clear from the above graph that $\lim_{x\to 3^-} f(x) = -\infty$ and $\lim_{x\to 3^+} f(x) = \infty$, hence $f(x)$ is not continuous at $x = 3$. Note that $f(x)$ is not even defined at $x = 3$.

2.1.4. One-sided continuity

Just like one-sided limits, we have one-sided continuity. A function $f(x)$ is called

- **Left Continuous** at $x = a$ if $\lim_{x\to a^-} f(x) = f(a)$
- **Right Continuous** at $x = a$ if $\lim_{x\to a^+} f(x) = f(a)$

Example 2.2. Discuss the continuity of

$$f(x) = \begin{cases} x & x < 2 \\ 2 & 2 \le x \le 5 \\ x^2 - 1 & x > 5 \end{cases}$$

Solution. The candidates for points of discontinuity are 2 and 5. For $x = 2$,

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} x = 2,$$
$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} 2 = 2 \quad \text{and}$$
$$f(2) = 2.$$

Therefore, $f(x)$ is continuous at $x = 2$. For $x = 5$,

$$\lim_{x\to 5^-} f(x) = \lim_{x\to 5^-} 2 = 2,$$

$$\lim_{x\to 5^+} f(x) = \lim_{x\to 5^+} x^2 - 1 = 24 \quad \text{and}$$

$$f(5) = 2.$$

Thus $\lim_{x\to 5^-} f(x) \neq \lim_{x\to 5^+} f(x)$, however, $\lim_{x\to 5^-} f(x) = f(5)$. Thus $f(x)$ is left-continuous at $x = 5$.

□

2.1.5. Standard continuous functions

2.1.5.1. **Basic laws of Continuity.** If $f(x)$ and $g(x)$ are continuous at $x = a$, then the following functions are also continuous at $x = a$.

(a) $f(x) + g(x)$ and $f(x) - g(x)$.
(b) $f(x)g(x)$.
(c) $kf(x)$ for any constant k.
(d) $f(x)/g(x)$ if $g(a) \neq 0$.

2.1.5.2. **Continuity of some basic functions.**

- $y = \sin x$ and $y = \cos x$ are continuous for every real number.
- $y = P(x)$, a polynomial is continuous for all real numbers.
- For polynomials $P(x)$ and $Q(x)$, the quotient $P(x)/Q(x)$ is continuous on its domain (at all values of $x = a$ such that $Q(a) \neq 0$).

Example 2.3. Discuss the continuity of the function

$$f(x) = \frac{x-1}{x^2-4}$$

Solution. As $x - 1$ and $x^2 - 4$ are both polynomials therefore, the function $f(x)$ is not continuous at points where the denominator is zero. Those points are $x = \pm 2$.

□

2.1.5.3. **Continuity of the Inverse Function.** If $f(x)$ is continuous on an interval I with range R, and if $f^{-1}(x)$ exists, then $f^{-1}(x)$ is continuous with domain R.

Example 2.4. Discuss the continuity of the function $f(x) = \sin^{-1} x$.

Solution. Since $y = \sin x$ is continuous on the real line with range $[-1, 1]$, therefore $y = \sin^{-1} x$ is continuous on the interval $[-1, 1]$.

□

2.1.5.4. **Continuity of Composite Functions.** If g is continuous at $x = a$, and f is continuous $x = g(a)$, then the composite function $f(x) = f(g(x))$ is continuous at $x = a$.

Example 2.5. Discuss the continuity of the function $F(x) = \sin\left(\frac{1}{x-1}\right)$.

Solution. The function $f(x) = \sin x$ is continuous for all real numbers and the function $g(x) = \frac{1}{x-1}$ is continuous for all real numbers except 1. Since $F(x) = f(g(x))$, therefore $F(x)$ is continuous for all real numbers except 1.

□

2.2. Evaluating Limits Algebraically

2.2.1. Performance Criteria

(a) Calculate limits at a point.
(b) Evaluate limits of indeterminate forms.
(c) Calculate left and right hand limits.

2.2.1.1. **Evaluating limits by substitution.** We can use substitution to evaluate limits of functions that are continuous at the point of substitution, i.e. $\lim_{x\to a} f(x) = L$, if $f(a) = L$ where L is a finite number.

Note that if $f(a)$ is undefined then the limit could still exist, it just means that we have to use other methods or use simplification in order to evaluate the corresponding limit.

Example 2.6. Evaluate the limit.

$$\lim_{x\to -2} \frac{x^2-9}{x+3}$$

Solution. Let $f(x) = \frac{x^2-9}{x+3}$. Substituting $x = -2$ in this function gives $f(-2) = \frac{(-2)^2-9}{-2+3} = -5$ which is a finite number. Hence,

$$\lim_{x\to -2} \frac{x^2-9}{x+3} = -5$$

□

2.2.1.2. **Evaluating indeterminate limits.** Most of the time the above substitution method does not work with limits. This is because when we substitute $x = a$, the function $f(x)$ takes one of the following forms

$$\frac{0}{0}, \quad \frac{\infty}{\infty}, \quad \infty \cdot 0, \quad \infty - \infty$$

In this situation we usually simplify the function $f(x)$ and, in the process get rid of the term that makes the function undefined.

Example 2.7. Evaluate

$$\lim_{x\to 5} \frac{x^2 - 25}{x - 5}$$

Solution. Let $f(x) = \dfrac{x^2 - 25}{x - 5}$. Substitution gives $f(5) = \dfrac{0}{0}$. Hence, let's try to simplify the function,

$$\begin{aligned}
\lim_{x\to 5} \frac{x^2 - 25}{x - 5} &= \lim_{x\to 5} \frac{(x-5)(x+5)}{x-5} \\
&= \lim_{x\to 5}(x+5) \text{ (We can divide by } x - 5 \text{ as } x \neq 5 \text{ but } x \to 5) \\
&= 5 + 5 = 10. \text{ (Now we can use substitution again)}
\end{aligned}$$

□

Example 2.8. Evaluate

$$\lim_{x\to -2} \frac{x^3 + 8}{x^2 + 6x + 8}$$

Solution. Let $f(x) = \dfrac{x^3 + 8}{x^2 + 6x + 8}$. Substitution gives $f(-2) = \dfrac{0}{0}$. Let's try to simplify the function,

$$\begin{aligned}
\lim_{x\to -2} \frac{x^3 + 8}{x^2 + 6x + 8} &= \lim_{x\to -2} \frac{(x+2)(x^2 - 2x + 4)}{(x+2)(x+4)} \text{ (Recall, } a^3 \pm b^3 = (a \pm b)(a^2 \mp ab + b^2)) \\
&= \lim_{x\to -2} \frac{x^2 - 2x + 4}{x + 4} \text{ (We can divide by } x + 2 \text{ as } x \neq -2 \text{ but } x \to -2) \\
&= \frac{(-2)^2 - 2(-2) + 4}{-2 + 4} = 6. \text{ (Now we can use substitution again)}
\end{aligned}$$

□

Example 2.9. Evaluate

$$\lim_{x\to 0} \frac{\cot x}{\csc x}$$

Solution. Let $f(x) = \dfrac{\cot x}{\csc x}$. Substitution gives $f(0) = \dfrac{\infty}{\infty}$. Therefore, let's try to simplify the function.

$$\begin{aligned}
\lim_{x\to 0} \frac{\cot x}{\csc x} &= \lim_{x\to 0} \frac{\frac{\cos x}{\sin x}}{\frac{1}{\sin x}} \\
&= \lim_{x\to 0} \cos x \text{ (We can divide by } \sin x \text{ as } \sin x \neq 0 \text{ since } x \neq 0 \text{ but } x \to 0) \\
&= \cos 0 = 1. \text{ (Now we can use substitution again)}
\end{aligned}$$

□

Example 2.10. Evaluate

$$\lim_{x\to 1} \left(\frac{1}{1-x} - \frac{2}{1-x^2} \right)$$

Solution. Let $f(x) = \left(\dfrac{1}{1-x} - \dfrac{2}{1-x^2} \right)$. Substitution gives $f(1) = \infty - \infty$. Hence, let's try to simplify the function,

$$\begin{aligned}
\lim_{x\to 1} \left(\frac{1}{1-x} - \frac{2}{1-x^2} \right) &= \lim_{x\to 1} \left(\frac{1+x-2}{1-x^2} \right) \\
&= \lim_{x\to 1} \left(\frac{x-1}{1-x^2} \right) \\
&= \lim_{x\to 1} \frac{x-1}{(1-x)(1+x)} \text{ (We can divide by } x-1 \text{ as } x \neq 1 \text{ but } x \to 1) \\
&= \lim_{x\to 1} \frac{-1}{1+x} \text{ (Now we can use substitution again)} \\
&= -\frac{1}{2}
\end{aligned}$$

□

Example 2.11. Evaluate

$$\lim_{x\to 8} \frac{\sqrt{x-4}-2}{x-8}$$

Solution. Let $f(x) = \dfrac{\sqrt{x-4}-2}{x-8}$. Substitution gives $f(8) = \dfrac{0}{0}$. Therefore, let's simplify the function. We first multiply the numerator and the denominator with the conjugate of the numerator.

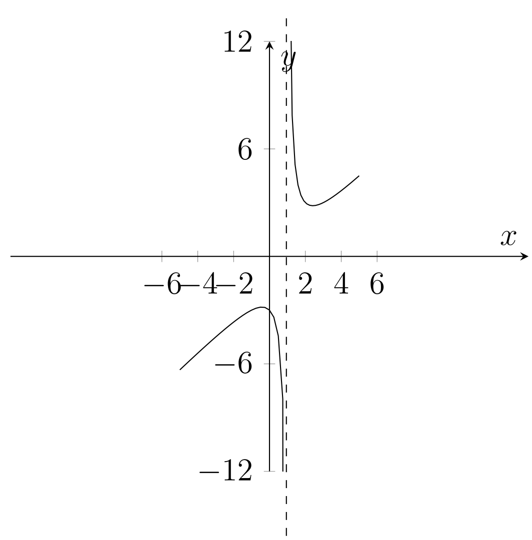
$$\begin{aligned}
\lim_{x\to 8} \frac{\sqrt{x-4}-2}{x-8} &= \lim_{x\to 8} \frac{(\sqrt{x-4}-2)(\sqrt{x-4}+2)}{(x-8)(\sqrt{x-4}+2)} \\
&= \lim_{x\to 8} \frac{x-4-4}{(x-8)(\sqrt{x-4}+2)} \\
&= \lim_{x\to 8} \frac{x-8}{(x-8)(\sqrt{x-4}+2)} \\
&= \lim_{x\to 8} \frac{1}{\sqrt{x-4}+2} \text{ (We can divide by } x-8 \text{ as } x \neq 8 \text{ but } x \to 8) \\
&= \frac{1}{\sqrt{8-4}+2} = \frac{1}{4} \text{ (We use substitution here)}
\end{aligned}$$

□

Example 2.12. Evaluate

$$\lim_{x\to 1} \frac{x^2-2x+3}{x-1}$$

Solution. Let $f(x) = \dfrac{x^2-2x+3}{x-1}$. Substitution gives $f(1) = \dfrac{2}{0}$. This is not an indeterminate form, so simplification won't help. Let's try to graph the function,



From the above graph, we see that the one-sided limits are infinite.

$$\lim_{x\to 1^-} \frac{x^2-2x+3}{x-1} = -\infty, \qquad \lim_{x\to 1^+} \frac{x^2-2x+3}{x-1} = \infty$$

Therefore the limit does not exist.

$\square$

Handout 4
MATH 251

Dibyajyoti Deb

4.1. Definition of the Derivative

4.1.1. Performance Criteria

(a) Derive a formula for the definition of the derivative.
(b) Use the definition of the derivative to calculate the derivative of a polynomial.
(c) Use the definition of the derivative to calculate the equation of the line tangent to a polynomial at a point.
(d) Use derivative notations correctly.

4.1.2. The Derivative

Differential calculus can be broadly classified into 3 main topics. They are

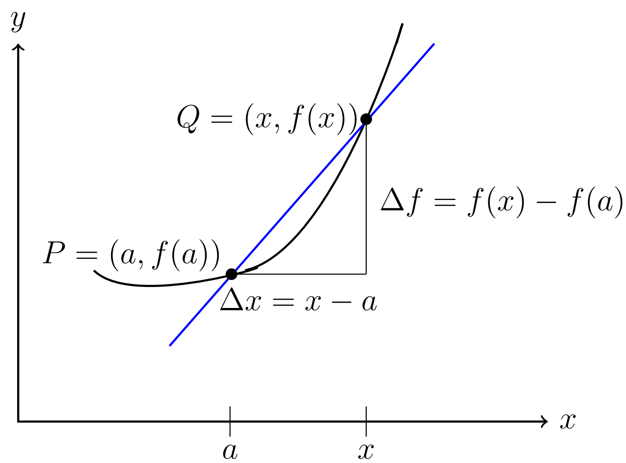
(a) The study of limits and continuity.
(b) The study of the derivative.
(c) Applications of the derivative.

We have already covered the first topic of limits and continuity in our previous 3 handouts. In this handout and beyond we will be looking at the second and the most important topic of the **derivative**.

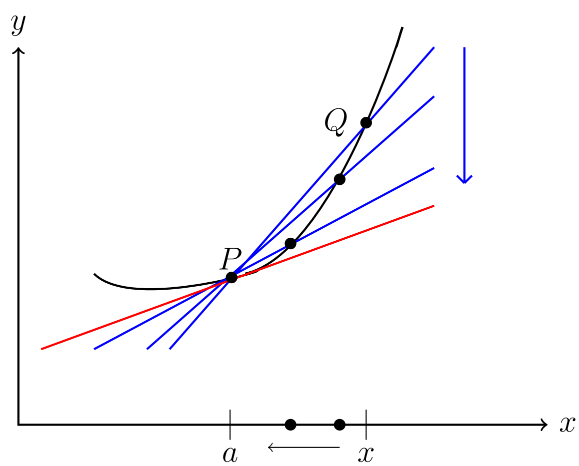
Let's go back to the topic of average rate of change that we covered earlier. I said that the average rate of change of a function $f(x)$ on the interval $[a, b]$ is given by the expression

$$\frac{\Delta f}{\Delta x} = \frac{f(x) - f(a)}{x - a}$$

So what exactly did we find out when we computed this **difference quotient**?



If we look at the graph above, we can see that the difference quotient is nothing but the **slope** of the secant line PQ. Now imagine what would happen if we keep the point P fixed, but keep on moving the point Q towards P.



The blue secant lines would eventually become the **tangent line** at the point P. In the process, the point x on the x-axis gets closer to a. Thus, we would expect the slope of the secant lines to approach the slope of the tangent line. This leads us to the definition of this new term called the **derivative**.

Definition 4.1. The **derivative** of $f(x)$ at $x = a$ is the limit of the difference quotient (if it exists)

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \tag{4.2}$$

When the limit exist, we say that f is **differentiable** at $x = a$.

Note that the derivative of $f(x)$ at $x = a$ is denoted by $f'(a)$, which is read as "f prime of a". We take the limit of the difference quotient in the definition of the derivative because as point Q approaches P, the point x approaches a which is signified by the limit.

Since the difference quotient is the slope of the secant lines joining P and the moving point Q, hence the above limit is also the slope of the tangent line at P since as Q approaches P (or x approaches a), the secant lines end up becoming the tangent line at P.

4.1.2.1. **Tangent Line.** Assume that $f(x)$ is differentiable at $x = a$. Then the slope of the tangent line to the graph $y = f(x)$ at the point $(a, f(a))$ is $f'(a)$ which is the derivative of $f(x)$ at $x = a$. Thus, the equation of the tangent line in point-slope form is

$$y - f(a) = f'(a)(x - a)$$

4.1.2.2. **Equivalent definition of the derivative.** Imagine that the distance between the points x and a is h. Then

$$x - a = h \qquad \text{or} \qquad x = a + h$$

We can then re-define our earlier limit. As x approaches a, the distance between them which is h gets smaller and smaller, therefore h approaches 0. Using this new variable h we have an equivalent definition of the derivative.

Definition 4.3. The **derivative** of $f(x)$ at $x = a$ is the limit of the quotient (if it exists)

$$f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h} \tag{4.4}$$

We will be using this new definition in most of the problems from this section (You can use the other definition too, you will get the same answer).

Example 4.5. Find $f'(2)$ using both Equations 4.4 and 4.2.

$$f(x) = x^2 + 9x$$

Solution. First let us use Equation 4.4. Here $a = 2$. Thus, $f(2+h) = (2+h)^2 + 9(2+h) = 4 + 4h + h^2 + 18 + 9h = h^2 + 13h + 22$. On the other hand, $f(2) = 2^2 + 9 \cdot 2 = 4 + 18 = 22$. By Equation 4.4,

$$\begin{aligned}
f'(2) &= \lim_{h\to 0} \frac{f(2+h) - f(2)}{h} \\
&= \lim_{h\to 0} \frac{(h^2 + 13h + 22) - 22}{h} \\
&= \lim_{h\to 0} \frac{h^2 + 13h}{h} \\
&= \lim_{h\to 0} \frac{h(h+13)}{h} \\
&= \lim_{h\to 0}(h + 13) = 13.
\end{aligned}$$

Now, let us use Equation 4.2. Here again $a = 2$. Thus by Equation 4.2,

$$\begin{aligned}
f'(2) &= \lim_{x\to 2} \frac{f(x) - f(2)}{x - 2} \\
&= \lim_{x\to 2} \frac{x^2 + 9x - 22}{x - 2} \\
&= \lim_{x\to 2} \frac{(x+11)(x-2)}{x - 2} \\
&= \lim_{x\to 2}(x + 11) = 2 + 11 = 13.
\end{aligned}$$

□

Example 4.6. Find the equation of the tangent line to the curve $y = f(x)$ at the specified point a.

$$f(x) = \sqrt{x+4}, \qquad a = 1$$

Solution. To find the equation of the tangent line we have to first find the slope of the tangent line at the point $x = 1$. As we have seen before the slope of the tangent line at $x = 1$ is $f'(1)$, hence let us find $f'(1)$

by using Equation 4.4,

$$\begin{aligned}
f'(1) &= \lim_{h\to 0} \frac{f(1+h)-f(1)}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{5+h}-\sqrt{5}}{h} \\
&= \lim_{h\to 0} \frac{(\sqrt{5+h}-\sqrt{5})}{h} \cdot \frac{(\sqrt{5+h}+\sqrt{5})}{(\sqrt{5+h}+\sqrt{5})} \\
&= \lim_{h\to 0} \frac{(5+h)-5}{h(\sqrt{5+h}+\sqrt{5})} \\
&= \lim_{h\to 0} \frac{h}{h(\sqrt{5+h}+\sqrt{5})} \\
&= \lim_{h\to 0} \frac{1}{\sqrt{5+h}+\sqrt{5}} = \frac{1}{\sqrt{5}+\sqrt{5}} = \frac{1}{2\sqrt{5}}
\end{aligned}$$

Thus, the slope of the tangent line is $\frac{1}{2\sqrt{5}}$. We can find the equation of the tangent line using the point slope form.

$$\begin{aligned}
y - f(1) &= \frac{1}{2\sqrt{5}}(x-1) \\
y - \sqrt{5} &= \frac{x}{2\sqrt{5}} - \frac{1}{2\sqrt{5}} \\
y &= \frac{x}{2\sqrt{5}} - \frac{1}{2\sqrt{5}} + \sqrt{5} \\
y &= \frac{x}{2\sqrt{5}} + \frac{9}{2\sqrt{5}}
\end{aligned}$$

□

4.2. The Derivative as a Function

4.2.1. Performance Criteria

(a) Use the derivative to calculate the derivative of a discrete function.
(b) Use derivative notations correctly.
(c) Use the graph of a function to draw the graph of the derivative.

4.2.2. The Derivative Function

In the previous section, we computed the derivative of $f(x)$ at a specific point a which we denoted by $f'(a)$. Now, we find the "general" definition of the derivative of the function $f(x)$. The definition stays mostly the same with the a replaced by the variable x.

Definition 4.7. The derivative of the function $f(x)$ is the new function denoted by $f'(x)$ and is the limit

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \tag{4.8}$$

if it exists. We say that $f(x)$ is **differentiable** on (a, b) if $f'(x)$ exists for all x in (a, b).

Example 4.9. Compute $f'(x)$ and find an equation of the tangent line at $x = 5$.

$$f(x) = \frac{1}{x^2}$$

Solution. Let us find $f'(x)$ first and then we will find $f'(5)$. By Equation 4.8,

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{\dfrac{1}{(x+h)^2} - \dfrac{1}{x^2}}{h} \\
&= \lim_{h\to 0} \frac{x^2 - (x+h)^2}{hx^2(x+h)^2} \\
&= \lim_{h\to 0} \frac{x^2 - (x^2 + 2xh + h^2)}{hx^2(x+h)^2} \\
&= \lim_{h\to 0} \frac{-2xh - h^2}{hx^2(x+h)^2} \\
&= \lim_{h\to 0} \frac{h(-2x - h)}{hx^2(x+h)^2} \\
&= \lim_{h\to 0} \frac{-2x - h}{x^2(x+h)^2} = \frac{-2x}{x^2 \cdot x^2} = -\frac{2}{x^3}
\end{aligned}$$

Therefore, $f'(5) = -\frac{2}{5^3} = -\frac{2}{125}$, which is the slope of the tangent line at $x = 5$. To find the equation of the tangent line, we use the point-slope form.

$$\begin{aligned} y - f(5) &= f'(5)(x-5) \\ y - \frac{1}{25} &= -\frac{1}{125}(x-5) \\ y - \frac{1}{25} &= -\frac{x}{125} + \frac{1}{25} \\ y &= -\frac{x}{125} + \frac{2}{25} \end{aligned}$$

□

4.2.3. Different notations for the derivative

We have already looked at $f'(x)$ which is one way to denote the derivative of $f(x)$. There is another standard notation for the derivative that is due to Leibnitz.

$$\frac{dy}{dx}$$

It is read as "derivative of y with respect to x", where y is the function $f(x)$. Remember, that you cannot "cancel" the d's in the notation. Sometimes the notation $\frac{df}{dx}$ is also used. Thus, from the previous example, if $y = \frac{1}{x^2}$, then

$$\frac{dy}{dx} = -\frac{2}{x^3} \quad \text{or} \quad \frac{d}{dx}(x^{-2}) = -\frac{2}{x^3}$$

To denote the value of the derivative at a specific point, say, $x = 5$, we write

$$\left.\frac{dy}{dx}\right|_{x=5} = -\frac{2}{125}$$

4.2.4. Some standard derivatives and Linearity rules

Since we now have a fairly decent idea of what a derivative is, therefore, let us look at some standard derivatives. We do not want to use the definition of the derivative involving limits all the time.

4.2.4.1. **The Power Rule.** For all exponents n,

(4.10) $$\frac{d}{dx}(x^n) = nx^{n-1}$$

Example 4.11. Find

$$\frac{d}{dx}(x^{-3/5})$$

Solution. By Equation 4.10, we have $n = -3/5$. Thus,

$$\frac{d}{dx}(x^{-3/5}) = -\frac{3}{5}x^{-\frac{3}{5}-1} = -\frac{3}{5}x^{-8/5}$$

□

4.2.4.2. **The derivative of e^x.** For the exponential constant e,

$$\frac{d}{dx}(e^x) = e^x$$

4.2.4.3. **Linearity Rules.** Assume that the functions $f(x)$ and $g(x)$ are differentiable. Then $f(x) \pm g(x)$ and $cf(x)$ for any constant c are differentiable and,

(a) $\frac{d}{dx}(f(x) \pm g(x)) = \frac{df(x)}{dx} \pm \frac{dg(x)}{dx}$

(b) $\frac{d}{dx}(cf(x)) = c\,\frac{df(x)}{dx}$

Example 4.12. Calculate $\left.\frac{df}{dt}\right|_{t=1}$, where $f(t) = 2t^{-2} + 3e^t - t^{-2/5}$.

Solution. Note that here the independent variable is t and so we are finding the derivative of f with respect to t. Using the linearity rules from above,

$$\begin{aligned}
\frac{df}{dt} &= 2\frac{d(t^{-2})}{dt} + 3\frac{d(e^t)}{dt} - \frac{d(t^{-2/5})}{dt} \\
&= 2(-2)t^{-2-1} + 3e^t - (-\frac{2}{5})t^{-\frac{2}{5}-1} \\
&= -4t^{-3} + 3e^t + \frac{2}{5}t^{-7/5}
\end{aligned}$$

□

Example 4.13. Find the points on the graph of $f(x) = 12x - x^3$ where the tangent is horizontal.

Solution. For the tangent to be horizontal the slope has to be equal to 0. Therefore, let us find the expression for the slope of the tangent by finding derivative function.

$$\begin{aligned} f'(x) = \frac{df}{dx} &= 12\frac{d(x)}{dx} - \frac{d(x^3)}{dx} \\ &= 12 \cdot 1x^{1-1} - 3x^{3-1} \\ &= 12 - 3x^2 \end{aligned}$$

The tangent is horizontal when

$$f'(x) = 12 - 3x^2 = 0 \Rightarrow x = \pm 2$$

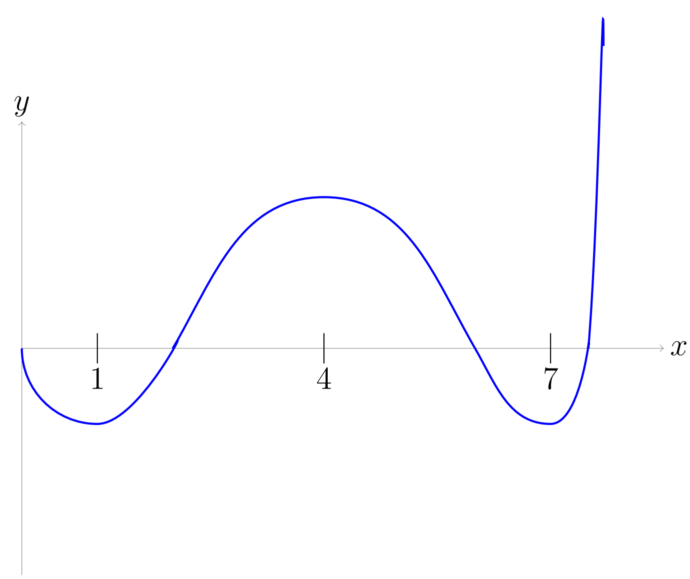
We find the y coordinates from $f(2) = 12 \cdot 2 - 2^3 = 16$ and $f(-2) = 12 \cdot (-2) - (-2)^3 = -16$. Thus the two points where the tangent is horizontal are $(2, 16)$ and $(-2, -16)$.

□

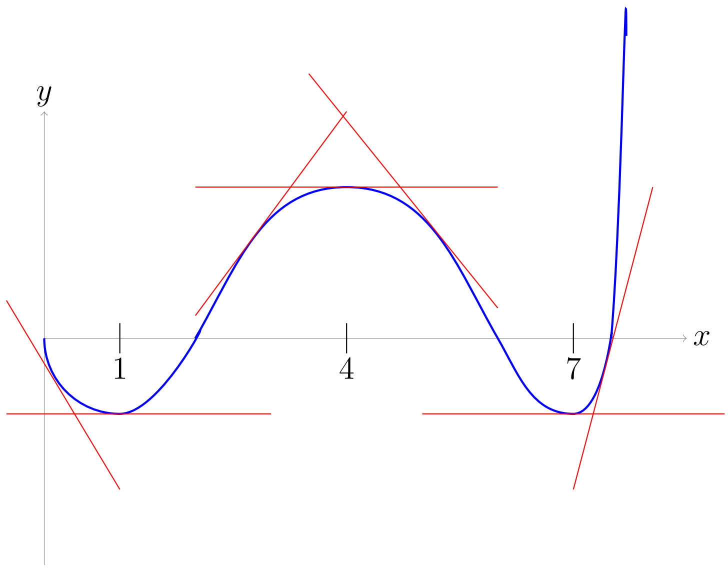
4.2.5. The graph of the derivative

Given the graph of the function, can we somehow sketch the graph of its derivative function? It can be done. Let's look at an example.

Example 4.14. The graph of $f(x)$ is shown below. Use it to sketch the graph of $f'(x)$.



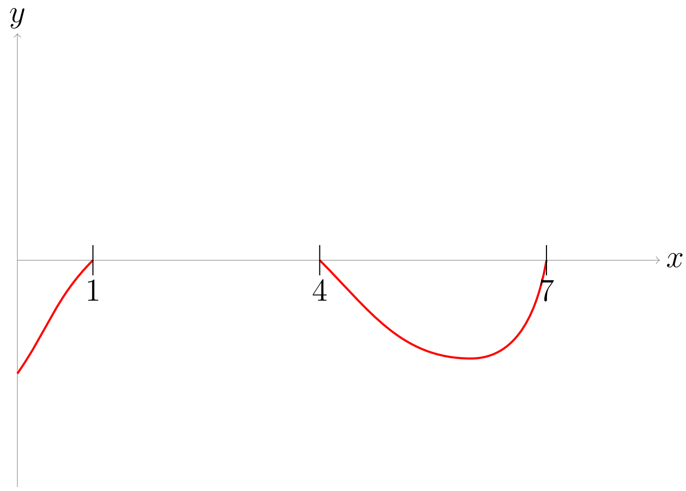
Solution. We draw tangent lines at various point on the graph by red lines.



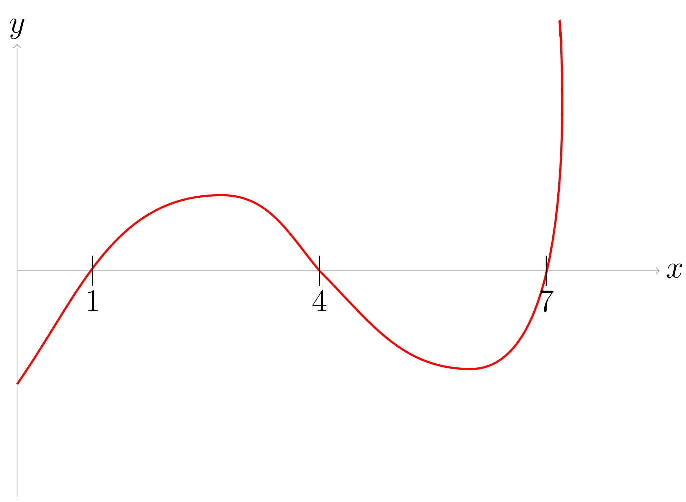
By looking at the shape of the tangent line at various intervals we have the following information.

Slope of the tangent line ($f'(x)$)	Interval
Negative	$(0, 1) \cup (4, 7)$
Positive	$(1, 4) \cup (7, \infty)$
Zero	$x = 1, 4, 7$

As we see from the table above that $f'(x)$ is negative on the intervals $(0, 1)$ and $(4, 7)$. We use this to sketch the graph of $f'(x)$ by drawing it below the x-axis on those intervals. Note, however, that we sketch the graph of $f'(x)$ on $(0, 1)$ to be increasing. This is because if we look at the changing shape of the tangent line on $(0, 1)$, we see that the slopes of the tangent lines are increasing. Similarly, even though the slope of the tangent line is negative on the interval $[4, 7]$, the slope is decreasing between 4 and approximately 6 and increasing thereafter until 7. Let us graph a partial section of the graph on the intervals where $f'(x)$ is negative.



The graph of $f'(x)$ crosses the x-axis at $x = 1, 4, 7$, since those are the points where the slope of the tangent line is zero. (the tangent is horizontal.) $f'(x)$ is positive on the intervals $(1, 4)$ and $(7, \infty)$. However, if we look at the shape of the tangent lines we see that the slopes of the tangent line is increasing between 1 and approximately 2.5, and starts to decrease thereafter until 4. (the slope is still positive everywhere on $[1, 4]$) Similarly, the slope of the tangent line keeps increasing after 7. If we put all this information together with the previous partial graph that we had, we end with the complete picture of the graph of $f'(x)$.



□

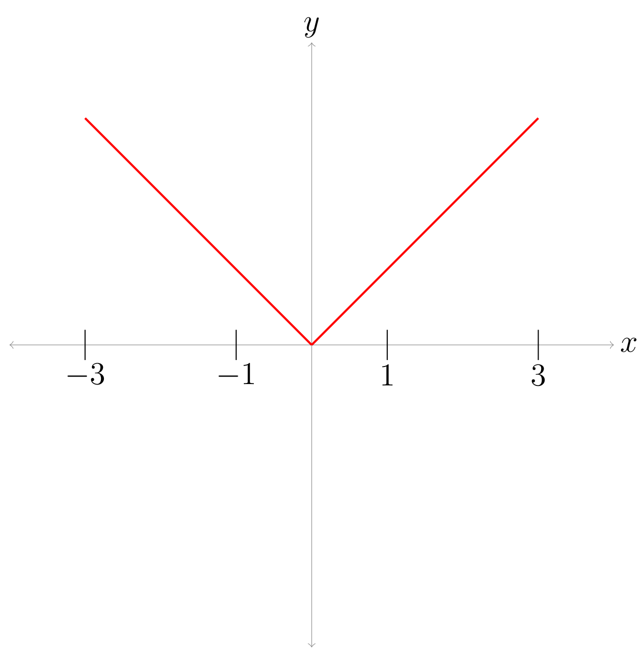
4.2.6. Relation between Differentiability and Continuity

We have already looked at one definition of differentiability before in terms of existence of a certain limit. (refer to Equations 4.2 and 4.4) Now let us look at an "informal" way of determining whether a function is differentiable at a certain point by looking at its graph.

If a graph of a function has a "break" or a "corner" at any

point then the function is not differentiable at that point.

We have already see what a "break" means on a graph when we looked at continuity before, but what does a "corner" mean? Let's look at the graph below.



This is the graph of the function $f(x) = |x|$. The graph doesn't seem to have any "break" anywhere, so informally it seems to be continuous everywhere. However the graph is not "smooth" at the origin. The graph has a "corner" there. It abruptly changes direction at the origin. Hence we can informally say that the function $f(x) = |x|$ is **not differentiable** at $x = 0$. Now let us prove the same fact rigorously,

Example 4.15. Show that $f(x) = |x|$ is continuous but not differentiable at $x = 0$.

Solution. The function $f(x)$ is continuous at $x = 0$ because $\lim_{x \to 0} |x| = 0 = f(0)$. However, for differentiability we need to check the existence of the limit below. By Equation 4.4,

$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$

The left hand limit is

$$\begin{aligned}
\lim_{h \to 0^-} \frac{f(0+h) - f(0)}{h} &= \lim_{h \to 0^-} \frac{f(h) - |0|}{h} \\
&= \lim_{h \to 0^-} \frac{|h|}{h} \\
&= \lim_{h \to 0^-} \frac{-h}{h} \text{ (Since } |h| = -h \text{ when } h < 0) \\
&= \lim_{h \to 0^-} -1 = -1.
\end{aligned}$$

whereas the right hand limit is

$$\begin{aligned}
\lim_{h \to 0^+} \frac{f(0+h) - f(0)}{h} &= \lim_{h \to 0^+} \frac{f(h) - |0|}{h} \\
&= \lim_{h \to 0^+} \frac{|h|}{h} \\
&= \lim_{h \to 0^+} \frac{h}{h} \text{ (Since } |h| = h \text{ when } h \geq 0) \\
&= \lim_{h \to 0^+} 1 = 1.
\end{aligned}$$

As we can see the left hand and the right hand limits are not the same, therefore

$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} \text{ does not exist}$$

Hence, $f(x) = |x|$ is not differentiable at $x = 0$.

□

Now let us look at an important relation between differentiability and continuity by means of this theorem.

Theorem 4.16. *If a function f is differentiable at $x = a$, then f is continuous at $x = a$, i.e.* **differentiability implies continuity**.

Note that the other way round is not true as we saw in the previous example. The function $f(x) = |x|$ was continuous at $x = 0$, but was not differentiable at $x = 0$. However, the contrapositive of the statement is true.

Corollary 4.17. If a function f is not continuous at $x = a$, then it is not differentiable at $x = a$.

4.3. Product and Quotient Rules

4.3.1. Performance Criteria

(a) Calculate explicit derivatives of functions of polynomials and exponential functions with the power, product and quotient rules.

4.3.2. Product Rule

In this section we learn another powerful tool by which we can compute the derivatives of more complicated functions which are written as products.

Definition 4.18. Product Rule If $f(x)$ and $g(x)$ are differentiable functions of x, then $f(x)g(x)$ is differentiable too and

$$\frac{d}{dx}(f(x)g(x)) = f(x)\frac{d(g(x))}{dx} + g(x)\frac{d(f(x))}{dx}$$

or in short,

$$(fg)' = fg' + gf'$$

Example 4.19. Find the derivative of $h(x) = (x^2 + 3x)(2x - 1)$.

Solution. Here we clearly see a product, so let $f(x) = x^2 + 3x$ and $g(x) = 2x - 1$. Then by the Product Rule,

$$\begin{aligned} h'(x) &= (x^2 + 3x)\frac{d(2x-1)}{dx} + (2x-1)\frac{d(x^2+3x)}{dx} \\ &= (x^2 + 3x) \cdot 2 + (2x-1)(2x+3) \\ &= 2x^2 + 6x + 4x^2 + 4x - 3 \\ &= 6x^2 + 10x - 3. \end{aligned}$$

□

Example 4.20. Find the derivative of $h(t) = \left(2t^2 + \frac{1}{t}\right)e^t$.

Solution. Here again we have a product so we let $f(t) = 2t^2 + \dfrac{1}{t}$ and $g(t) = e^t$. Then by the Product Rule,

$$\begin{aligned} h'(t) &= \Big(2t^2 + \frac{1}{t}\Big)\frac{d(e^t)}{dt} + e^t\frac{d(2t^2 + \frac{1}{t})}{dt} \\ &= \Big(2t^2 + \frac{1}{t}\Big)e^t + e^t\Big(4t - \frac{1}{t^2}\Big) \\ &= 2t^2e^t + \frac{e^t}{t} + 4te^t - \frac{e^t}{t^2} \\ &= e^t\Big(2t^2 + \frac{1}{t} + 4t - \frac{1}{t^2}\Big) \end{aligned}$$

□

4.3.3. Quotient Rule

Now we learn another powerful tool by which we can compute the derivatives of more complicated functions which are written as quotients.

Definition 4.21. Quotient Rule If $f(x)$ and $g(x)$ are differentiable functions of x, then $\dfrac{f(x)}{g(x)}$ is differentiable too for all x such that $g(x) \neq 0$, and

$$\frac{d}{dx}\Big(\frac{f(x)}{g(x)}\Big) = \frac{g(x)\dfrac{d(f(x))}{dx} - f(x)\dfrac{d(g(x))}{dx}}{g(x)^2}$$

or in short,

$$\Big(\frac{f}{g}\Big)' = \frac{gf' - fg'}{g^2}$$

The important things to note here are the terms in the numerator and the denominator of the derivative. They have to be in that same exact form.

Example 4.22. Compute the derivative of $h(x) = \dfrac{2x}{x^3 - 1}$.

Solution. The function is a quotient, hence we let $f(x) = 2x$ and $g(x) = x^3 - 1$. By the Quotient Rule,

$$\begin{aligned} h'(x) &= \frac{(x^3-1)\dfrac{d(2x)}{dx} - 2x\dfrac{d(x^3-1)}{dx}}{(x^3-1)^2} \\ &= \frac{(x^3-1)\cdot 2 - 2x(3x^2)}{(x^3-1)^2} \\ &= \frac{2x^3 - 2 - 6x^3}{(x^3-1)^2} \\ &= \frac{-4x^3 - 2}{(x^3-1)^2} \end{aligned}$$

□

Example 4.23. Find $h'(2)$, where

$$h(x) = \frac{x^2 + e^x}{x+1}$$

Solution. Let us first find $h'(x)$. Since $h(x)$ is a quotient, therefore let $f(x) = x^2 + e^x$ and $g(x) = x + 1$. By the Quotient Rule,

$$\begin{aligned} h'(x) &= \frac{(x+1)\dfrac{d(x^2+e^x)}{dx} - (x^2+e^x)\dfrac{d(x+1)}{dx}}{(x+1)^2} \\ &= \frac{(x+1)(2x+e^x) - (x^2+e^x)\cdot 1}{(x+1)^2} \\ &= \frac{2x^2 + xe^x + 2x + e^x - x^2 - e^x}{(x+1)^2} \\ &= \frac{x^2 + x(2+e^x)}{(x+1)^2} \end{aligned}$$

Therefore,

$$h'(2) = \frac{2^2 + 2(2+e^2)}{(2+1)^2} = \frac{8+2e^2}{9}$$

□

Handout 6
MATH 251

Dibyajyoti Deb

6.1. Trigonometric Functions

6.1.1. Performance Criteria

(a) Calculate explicit derivatives of trigonometric functions.

6.1.2. Derivatives of trigonometric functions

We have already seen different techniques of differentiation which led us to finding derivatives of some complicated functions. We have also looked at some standard functions and their derivatives. In this section we look at some more standard functions and learn their derivatives and use them to find the derivatives of even more complicated functions.

6.1.2.1. **Derivatives of Sine and Cosine.** The derivatives of $\sin x$ and $\cos x$ are as follows.

Theorem 6.1. *The functions $y = \sin x$ and $y = \cos x$ are differentiable and,*

$$\frac{d}{dx}\sin x = \cos x \qquad \textit{and} \qquad \frac{d}{dx}\cos x = -\sin x$$

Example 6.2. Find the derivative of the function $f(x) = e^x \sin x$.

Solution. We use the product rule in this case with our two functions $g(x) = e^x$ and $h(x) = \sin x$. Recall that the product rule states that if g and h are functions of x, then

$$(gh)' = g'h + gh'$$

Therefore,

$$\begin{aligned} f'(x) &= \sin x \frac{d}{dx}(e^x) + e^x \frac{d}{dx}(\sin x) \\ &= \sin x \cdot e^x + e^x \cdot \cos x \\ &= e^x(\sin x + \cos x) \end{aligned}$$

□

6.1.2.2. **Derivatives of additional standard trigonometric functions.** We now look at the remaining standard trigonometric functions and their derivatives.

Theorem 6.3. *The functions* $y = \tan x$, $y = \sec x$, $y = \csc x$ *and* $y = \cot x$ *are differentiable and,*

$$\frac{d}{dx}\tan x = \sec^2 x \qquad \frac{d}{dx}\sec x = \sec x \tan x$$

$$\frac{d}{dx}\cot x = -\csc^2 x \qquad \frac{d}{dx}\csc x = -\csc x \cot x$$

Example 6.4. Find the derivative of the function $f(x) = \dfrac{1+\tan x}{1-\tan x}$.

Solution. We use the quotient rule in this case with our two functions $g(x) = 1 + \tan x$ and $h(x) = 1 - \tan x$. Recall that the quotient rule states that if g and h are functions of x, then

$$\left(\frac{g}{h}\right)' = \frac{hg' - gh'}{h^2}$$

Therefore,

$$\begin{aligned} f'(x) &= \frac{(1-\tan x)\frac{d}{dx}(1+\tan x) - (1+\tan x)\frac{d}{dx}(1-\tan x)}{(1-\tan x)^2} \\ &= \frac{(1-\tan x)\sec^2 x - (1+\tan x)(-\sec^2 x)}{(1-\tan x)^2} \\ &= \frac{\sec^2 x - \tan x \sec^2 x + \sec^2 x + \tan x \sec^2 x}{(1-\tan x)^2} \\ &= \frac{2\sec^2 x}{(1-\tan x)^2} \end{aligned}$$

□

Example 6.5. Find an equation of the tangent line to the curve $y = \csc x - \cot x$ at the point $x = \dfrac{\pi}{4}$.

Solution. To find the equation of the tangent line at a point we have to find the slope of the tangent line at that point first. This amounts to finding the derivative of $f(x) = \csc x - \cot x$ at the point $x = \frac{\pi}{4}$.

$$\begin{aligned} f(x) &= \csc x - \cot x \\ f'(x) &= -\csc x \cot x - (-\csc^2 x) \\ &= -\csc x \cot x + \csc^2 x \\ \text{Therefore, } f'(\frac{\pi}{4}) &= -\csc(\frac{\pi}{4})\cot(\frac{\pi}{4}) + \csc^2(\frac{\pi}{4}) \\ &= -\sqrt{2}\cdot 1 + (\sqrt{2})^2 \\ &= -\sqrt{2} + 2. \end{aligned}$$

We find the y-coordinate of the point by finding

$$f(\frac{\pi}{4}) = \csc(\frac{\pi}{4}) - \cot(\frac{\pi}{4}) = \sqrt{2} - 1.$$

We use the point-slope form with slope $m = 2 - \sqrt{2}$ and point $(\dfrac{\pi}{4}, \sqrt{2} - 1)$ to find the equation of the tangent line.

$$\begin{aligned} y - (\sqrt{2} - 1) &= (2 - \sqrt{2})(x - \frac{\pi}{4}) \\ y &= (2 - \sqrt{2})x - (2 - \sqrt{2})\frac{\pi}{4} + (\sqrt{2} - 1) \end{aligned}$$

□

6.2. The Chain Rule

6.2.1. Performance Criteria

(a) Calculate explicit derivatives of functions of polynomials, trigonometric functions, exponential, and logarithmic functions with the power, quotient, product, and chain rule.

6.2.2. The Chain Rule

In this section, we finally look at the most powerful technique that we will learn in the context of differentiation. Before we start looking at it, let us recall the definition of a **composite function**.

Given two function f and g of x, the composite of f and g is denoted by $f \circ g$ and is defined to be

$$(f \circ g)(x) = f(g(x))$$

So, for example if $h(x) = \sin(x^4)$, then we can say the $h(x)$ is the composite function $f(g(x))$ where $f(x) = \sin x$ and $g(x) = x^4$. Here we can also call $f(x)$ the "outer function" and $g(x)$ the "inner function".

Chain Rule shows us a way of finding the derivative of a composite function.

Theorem 6.6. Chain Rule *If f and g are differentiable functions of x, then the composite function $(f \circ g)(x) = f(g(x))$ is differentiable and*

$$(f(g(x))' = f'(g(x))g'(x)$$

In other words, when we want to use the chain rule to find the derivative of a function $h(x)$, our steps should include

1. Identifying the function given as a composition $f(g(x))$ along with the "outer function", $f(x)$ and the "inner function" $g(x)$.
2. Find $f'(x)$
3. Find $f'(g(x))$.
4. Find $g'(x)$.
5. Write $h'(x) = f'(g(x))g'(x)$.

Example 6.7. Find the derivative of the function $y = \sin(x^4)$.

Solution. We recognize that $\sin(x^4)$ can be written as the composite function $f(g(x))$ where $f(x) = \sin x$ and $g(x) = x^4$. We find the derivative of the "outer" function $\sin x$, which is $f'(x) = \cos x$ and the derivative of the "inner" function x^4, which is $g'(x) = 4x^3$. Now,

$$f'(g(x)) = f'(x^4) = \cos(x^4)$$

Therefore,

$$\frac{dy}{dx} = f'(g(x))g'(x) = \cos(x^4) \cdot 4x^3$$

□

We can also rewrite the chain rule using Leibniz notation.

> If $y = f(u)$, and $u = g(x)$, then
> $$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

Example 6.8. Calculate the derivative of $y = e^{2x^2}$.

Solution. Here we identify the "outer" function $f(x) = e^x$ and the "inner" function $g(x) = 2x^2$. Therefore, $f'(x) = e^x$ and $g'(x) = 4x$. Now $f'(g(x)) = e^{2x^2}$. Hence,

$$\frac{dy}{dx} = f'(g(x))g'(x) = e^{2x^2} \cdot 4x$$

□

Example 6.9. An expanding sphere has radius $r = 0.4t$ cm at time t (in seconds). Let V be the sphere's volume. Find dV/dt when (a) $r = 3$ and (b) $t = 3$.

Solution. The volume of a sphere of radius r is given by

$$V = \frac{4}{3}\pi r^3$$

We see here that V is a function of r and r is a function of t and we are asked to find dV/dt which is the derivative of V with respect to t. By the Chain Rule,

$$\frac{dV}{dt} = \frac{dV}{dr}\frac{dr}{dt}$$

We see that

$$\frac{dV}{dr} = 4\pi r^2 \qquad \text{and} \qquad \frac{dr}{dt} = 0.4$$

Therefore,

$$\frac{dV}{dt} = 4\pi r^2 \cdot 0.4 = 1.6\pi r^2$$

Now

$$\left.\frac{dV}{dt}\right|_{r=3} = 1.6\pi \cdot 3^2 = 14.4\pi \text{ cc/sec}$$

Since dV/dt is a function of r, therefore we need to find the value of r when $t = 3$. Using the relation between r and t,

$$r = 0.4t = 0.4 \cdot 3 = 1.2 \text{ cm}$$

Therefore,

$$\left.\frac{dV}{dt}\right|_{r=1.2} = 1.6\pi \cdot (1.2)^2 = 2.304\pi \text{ cc/sec}$$

□

Handout 5
MATH 251

Dibyajyoti Deb

5.1. Rates of Change

5.1.1. Performance Criteria

(a) Determine instantaneous rate of change from the derivative.
(b) Use derivative to find marginal cost and marginal profit.
(c) Given a position function, determine the velocity and acceleration for a particle in rectilinear motion.
(d) Solve problems involving motion under the influence of gravity.

5.1.2. Different rates of change

In this section we look at some applications of the derivative function that we learnt before. First, we look at the relation between the derivative and the instantaneous rate of change. Recall the definition of the average rate of change from before. If,

$$\Delta y = \text{ Change in } y = f(x) - f(a) \text{ and}$$

$$\Delta x = \text{ Change in } x = x - a$$

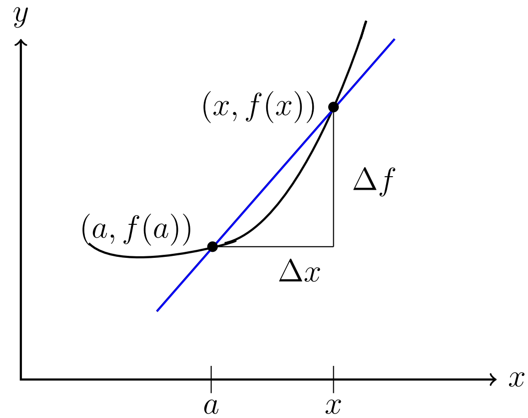
then,

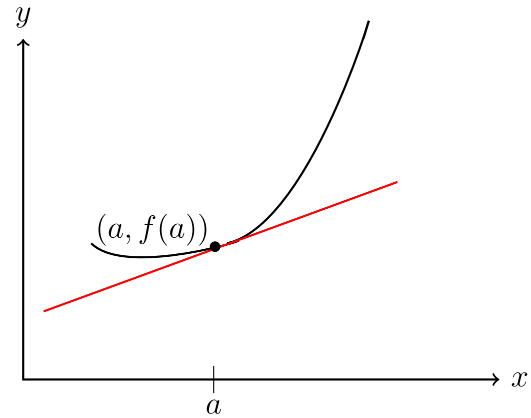
$$\text{Average rate of change } = \frac{\Delta y}{\Delta x} = \frac{f(x) - f(a)}{x - a}$$

Now recall the way we found the instantaneous rate of change. We did that by computing the average rate of change over very small intervals. This mean as x gets closer to a the difference quotient from above approaches the instantaneous rate of change at $x = a$. Therefore, we can define the **instantaneous rate of change** as

$$\text{Instantaneous rate of change at } x = a = f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

As far as the geometric interpretations go, the average rate of change is the slope of the secant line (blue) joining the points $(x, f(x))$ and $(a, f(a))$ and the instantaneous rate of change is the slope of the tangent line (red) at the point $(a, f(a))$ as shown below.





Example 5.1. Find the rate of change of the volume of a cube with respect to its side s when $s = 5$.

Solution. For a cube of side s, its volume $V(s)$ is s^3. The question asks us to find the rate of change at the point $s = 5$, which means that we have to find the instantaneous rate of change at that point. By our earlier definition, the (instantaneous) rate of change of V when $s = 5$ is given by,

$$V'(5) = \left.\frac{dV}{ds}\right|_{s=5}$$

Now

$$\begin{aligned} V(s) &= s^3 \\ V'(s) &= 3s^2 \\ V'(5) &= 3 \cdot 5^2 = 75 \text{ cubic units per unit} \end{aligned}$$

□

Looking at the previous problem, what exactly does $V'(5)$ mean? By using the definition of the derivative at a point we see that $V'(5) = 75$ is equivalent to saying that if we increase the length of each side of the cube from 5 by a very small amount, then the corresponding change in the volume of the cube would be close to 75 cubic units.

5.1.3. Effect of a One-Unit Change

We know that for small values of h (instead of saying $h \to 0$), the difference quotient is close to the derivative itself, i.e.

$$f'(a) \approx \frac{f(a+h) - f(a)}{h} \tag{5.2}$$

In some cases, h need not be very small rather $h = 1$ is enough for a good approximation. Therefore, substituting $h = 1$ in Equation 5.2 we have

$$f'(a) \approx f(a+1) - f(a) \tag{5.3}$$

Hence, $f'(a)$ *is approximately equal to the change of* f *caused by a one-unit change in* x *when* $x = a$.

5.1.3.1. **Marginal Cost in Economics.** One of the important applications of the above approximation is in economics where we find the marginal cost of an item. The **marginal cost** at a certain production level is the cost of producing one additional unit.

Thus, if $C(x)$ denote the cost of producing x units of a particular product, then the number of x units manufactured is called the **production level**. Hence, the marginal cost at production level a is given by

$$\text{Marginal cost } = C(a+1) - C(a)$$

Here Equation 5.3 usually gives a decent approximation, so we can use $C'(a)$ as an estimate for the marginal cost.

Example 5.4. The dollar cost of producing x donuts is $C(x) = 300 + 0.25x - 0.5(x/1000)^3$. Determine the cost of producing 2000 donuts and estimate the cost of the 2001st donut. Compare your estimate with the actual cost of the 2001st donut.

Solution. The cost to produce 2000 donuts is

$$C(2000) = 300 + 0.25(2000) - 0.5(2000/1000)^3 = \$796$$

The derivative is $C'(x) = 0.25 - 1.5x^2/1000^3$. We can estimate the marginal cost (cost to produce 1 additional item at current production level) at $x = 2000$ by the derivative

$$C'(2000) = 0.25 - 1.5(2000)^2/1000^3 = \$0.244.$$

Actual cost of producing 2001 donuts is

$$C(2001) = 300 + 0.25(2001) - 0.5(2001/1000)^3 = \$796.24399$$

Hence, the difference in the actual cost between production of 2000 and 2001 donuts is \$796.24399 − \$796 = \$0.24399 which is very close to the approximation of the marginal cost.

□

5.1.4. Linear Motion

Linear motion is the motion of a particle along a straight line. If $s(t)$ denotes the position or distance of a particle from the origin at time t, then

$$v(t) = \text{ Velocity } = \frac{ds}{dt}$$

Thus, **Velocity** is the rate of change of position with respect to time. **Speed** is defined as the absolute value of the velocity.

$$\text{Speed } = |v(t)| = \left|\frac{ds}{dt}\right|$$

Example 5.5. A particle moving along a line has position $s(t) = t^4 - 18t^2$ m at time t seconds. At which times does the particle pass through the origin? At which times is the particle instantaneously motionless (that is, it has zero velocity)?

Solution. Since $s(t)$ denotes the distance of the particle from the origin, therefore the particle passes through the origin when $s(t) = 0$. Solving for t we have,

$$\begin{aligned} t^4 - 18t^2 &= 0 \\ t^2(t^2 - 18) &= 0 \end{aligned}$$

$$t^2 = 0 \text{ and } t^2 = 18$$

This implies that the particle passes through the origin at times $t = 0$ and $t = \sqrt{18} = 3\sqrt{2}$ seconds.

To find the times when the particle has zero velocity, we find the expression for the velocity.

$$v(t) = \frac{ds}{dt} = 4t^3 - 36t$$

Equating $v(t) = 0$ we have,

$$\begin{aligned} 4t^3 - 36t &= 0 \\ 4t(t^2 - 9) &= 0 \end{aligned}$$

$$4t = 0 \text{ and } t^2 = 9$$

This implies that the particle has zero velocity at time $t = 0$ and $t = \sqrt{9} = 3$ seconds.

□

5.1.5. Motion under the influence of Gravity

Now we look at motion under the influence of gravity. This could be a falling object or an object tossed vertically up. Galileo discovered that the height $s(t)$ of an object rising or falling under the influence of gravity near the earth's surface is given by

$$s(t) = s_0 + v_0 t - \frac{1}{2} g t^2 \tag{5.6}$$

where s_0 is the initial position, v_0 is the initial velocity and g is the acceleration due to gravity ($g \approx 9.8$ m/s^2 or 32 ft/s^2). We can find the velocity of this object by finding the derivative,

$$v(t) = \frac{ds}{dt} = v_0 - gt \tag{5.7}$$

Note that the maximum height is attained by an object tossed vertically up when $v(t) = 0$.

Example 5.8. A ball tossed in the air vertically from ground level returns to earth 6 s later. Find the initial velocity and maximum height of the ball.

Solution. Since the ball is tossed from ground level, therefore the initial position of the ball $s_0 = 0$. After 6 seconds the ball hits the ground, therefore $s(6) = 0$. Using Equation 5.6 we have

$$\begin{aligned} s(6) &= 0 + 6v_0 - \frac{1}{2}(9.8)(6^2) \\ 0 &= 6v_0 - 176.4 \\ 6v_0 &= 176.4 \\ v_0 &= 29.4 \text{ m/s.} \end{aligned}$$

To find the maximum height, we find the time when $v(t) = 0$. Now by Equation 5.7,

$$v(t) = v_0 - gt = 29.4 - 9.8t$$

Setting it equal to zero we have,

$$\begin{aligned} 29.4 - 9.8t &= 0 \\ 9.8t &= 29.4 \\ t &= 3 \text{ seconds.} \end{aligned}$$

Therefore, the height reached after 3 seconds is

$$s(3) = 29.4(3) - \frac{1}{2}(9.8)3^2 = 44.1 \text{ m}.$$

□

5.2. Higher Derivatives

5.2.1. Performance Criteria

(a) Calculate higher order derivatives.
(b) Given a position function, determine the velocity and acceleration for a particle in rectilinear motion.

5.2.2. Higher order derivatives

We have already learned how to find derivatives of most functions using different rules. Now we look at how we can compute **higher derivatives**. Higher derivatives are obtained by repeatedly differentiating a function $y = f(x)$.

Thus, if $f(x)$ is differentiable, then the **first derivative** (which is the usual derivative) is denoted by $\frac{dy}{dx}$ or $f'(x)$. Now if f' is differentiable, then the **second derivative**, is the derivative of $f'(x)$ and is denoted by

$$f''(x) = \frac{d^2y}{dx^2}$$

So what exactly does $\frac{d^2y}{dx^2}$ mean?

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

Again, don't think that we are multiplying $\frac{d}{dx}$ with $\frac{dy}{dx}$ here. We are finding the derivative of the function $\frac{dy}{dx}$ with respect to x. Thus, continuing this way we can say that the nth derivative of f, provided that f is n-differentiable (i.e. all the previous $n-1$ derivatives exist) is denoted by

$$f^{(n)}(x) = \frac{d^ny}{dx^n}$$

Example 5.9. Calculate $f^{(4)}(1)$ for $f(x) = 4x^6 - 3x^5 + 7x^4 - 9x^3 + 2x^2 - 10$.

Solution. Let's find all the four derivative of f.

$$\begin{aligned}
f(x) &= 4x^6 - 3x^5 + 7x^4 - 9x^3 + 2x^2 - 10 \\
f'(x) &= \frac{d}{dx}(4x^6 - 3x^5 + 7x^4 - 9x^3 + 2x^2 - 10) = 24x^5 - 15x^4 + 28x^3 - 27x^2 + 4x \\
f''(x) &= \frac{d}{dx}(24x^5 - 15x^4 + 28x^3 - 27x^2 + 4x) = 120x^4 - 60x^3 + 84x^2 - 54x + 4 \\
f'''(x) &= \frac{d}{dx}(120x^4 - 60x^3 + 84x^2 - 54x + 4) = 480x^3 - 180x^2 + 168x - 54 \\
f^{(4)}(x) &= \frac{d}{dx}(480x^3 - 180x^2 + 168x - 54) = 1440x^2 - 360x + 168
\end{aligned}$$

Therefore,

$$f^{(4)}(1) = 1440 \cdot 1^2 - 360 \cdot 1 + 168 = 1248.$$

□

5.2.3. Acceleration

We have already seen the relationship between the position function $s(t)$ and the velocity function $v(t)$ earlier. Velocity is the rate of change of position with respect to time. Hence,

$$v(t) = \frac{ds}{dt} = s'(t)$$

A natural question to ask is, what is the average rate of change of velocity with respect to time? This is called the **acceleration** of a particle at time t and is denoted by $a(t)$. Hence,

$$a(t) = \frac{dv}{dt} = v'(t) = \frac{d^2s}{dt^2} = s''(t)$$

So, what does it mean for $a(t) = 0$? Since $a(t)$ is the derivative of $v(t)$, therefore $v(t)$ must be constant in order for its derivative to be zero, which means that the particle is travelling at a constant speed.

Example 5.10. Find the acceleration $a(t)$ of a ball tossed vertically in the air from the ground level with an initial velocity of 15 m/s.

Solution. By Galileo's formula (Equation 5.6), the height of the ball at time t is given by

$$s(t) = s_0 + v_0 t - \frac{1}{2}gt^2$$

For this problem $s_0 = 0$ (since it is tossed from the ground), $v_0 = 15$ (since that is the initial velocity) and $g = 9.8$, thus

$$s(t) = 15t - 4.9t^2$$

Therefore,

$$s'(t) = v(t) = 15 - 9.8t$$

and

$$v'(t) = a(t) = -9.8 \text{ m/s}^2$$

As expected, the acceleration is constant with value $-g = -9.8$ m/s^2. As the ball rises and falls, its velocity changes from 15 to -15 m/s at the constant rate $-g$.

$\square$

Handout 7
MATH 251

Dibyajyoti Deb

7.1. Derivatives of Inverse Functions

7.1.1. Performance Criteria

(a) Calculate explicit derivatives of inverse trigonometric functions.

7.1.2. Derivative of the Inverse

In this section we learn about the derivatives of inverse trigonometric functions. Before we jump into that, let's recap some facts about a function and its inverse.

7.1.2.1. **Inverse Functions.** When a function $f(x)$ is one-one, then it is invertible i.e. it has an inverse. We usually denote the inverse function by $f^{-1}(x)$ (Note that this is not the reciprocal of f). The function f and its inverse f^{-1} have the property

$$f(a) = b \Longleftrightarrow f^{-1}(b) = a$$

The two most important properties of a function f and its inverse f^{-1} are

$$f(f^{-1}(x)) = x \quad \text{and} \quad f^{-1}(f(x)) = x$$

Our goal now is to find the derivative of the inverse function i.e. $(f^{-1}(x))'$. We use one of the above compositions in order to do that.

$$f(f^{-1}(x)) = x$$

Taking derivatives on both sides and applying Chain Rule on the left we have

$$\begin{aligned} f'(f^{-1}(x))(f^{-1}(x))' &= 1 \\ (f^{-1}(x))' &= \frac{1}{f'(f^{-1}(x))} \end{aligned}$$

Thus, for $a \in$ Domain (f^{-1}),

$$(f^{-1}(x))'\Big|_{x=a} = \frac{1}{f'(f^{-1}(a))} \tag{7.1}$$

Example 7.2. Calculate $g'(-2)$, where g is the inverse of f.

$$f(x) = 4x^3 - 2x$$

Solution. By Equation 7.1, we have,

$$(g(x))'\Big|_{x=-2} = \frac{1}{f'(g(-2))}$$

Since g is the inverse of f, therefore if,

$$g(-2) = a \qquad \text{then} \qquad f(a) = -2$$

If we set $f(a) = -2$ then we have,

$$4a^3 - 2a = -2$$

which if we solve by guessing a small value of a (this is how we have to do it here since solving a cubic equation is harder than normal), we get $a = -1$.

Therefore, since $f(-1) = -2$, hence $g(-2) = -1$. Also,

$$f'(x) = 12x^2 - 2$$

and

$$f'(g(-2)) = f'(-1) = 12(-1)^2 - 2 = 10$$

Therefore,

$$(g(x))'\Big|_{x=-2} = \frac{1}{f'(g(-2))} = \frac{1}{10}$$

□

7.1.3. Derivatives of Inverse Trigonometric Functions

We look at some standard derivatives of inverse trigonometric functions in this section. These coupled with all the rules that we have learned before will help us in evaluating the derivatives of more complicated functions.

Theorem 7.3. *The derivatives of the inverse trigonometric functions $y = \sin^{-1} x$, $y = \cos^{-1} x$, $y = \tan^{-1} x$, $y = \sec^{-1} x$, $y = \csc^{-1} x$ and $y = \cot^{-1} x$ are as follows,*

$$\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\cos^{-1} x = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2} \qquad \frac{d}{dx}\sec^{-1}x = \frac{1}{|x|\sqrt{x^2-1}}$$
$$\frac{d}{dx}\cot^{-1}x = -\frac{1}{1+x^2} \qquad \frac{d}{dx}\csc^{-1}x = -\frac{1}{|x|\sqrt{x^2-1}}$$

Example 7.4. Find the derivative of $f(x) = (\tan^{-1}x)^3$.

Solution. We use the Chain Rule with the outer function x^3 and the inner function $\tan^{-1}x$ since their composition gives f. The derivative of x^3 at $\tan^{-1}x$ is $3(\tan^{-1}x)^2$ and the derivative of $\tan^{-1}x$ is $\frac{1}{1+x^2}$. Hence, by the Chain Rule,

$$f'(x) = \frac{3(\tan^{-1}x)^2}{1+x^2}$$

□

Example 7.5. Find the derivative of $f(x) = e^{\cos^{-1}x}$.

Solution. We use the Chain Rule with the outer function e^x and the inner function $\cos^{-1}x$ since their composition gives f. The derivative of e^x at $\cos^{-1}x$ is $e^{\cos^{-1}x}$ and the derivative of $\cos^{-1}x$ is $-\frac{1}{\sqrt{1-x^2}}$. Hence, by the Chain Rule,

$$f'(x) = -\frac{e^{\cos^{-1}x}}{\sqrt{1-x^2}}$$

□

7.2. Derivatives of General Exponential and Logarithmic Functions

7.2.1. Performance Criteria

(a) Calculate explicit derivatives of exponential and logarithmic functions with power, quotient, product and chain rule.
(b) Use Logarithmic differentiation to calculate the derivative of certain functions.

7.2.2. Derivative of the Exponential Function

An exponential function is of the form

$$f(x) = a^x, \quad a > 0, a \neq 1$$

The derivative of the exponential function is

$$\frac{d}{dx}(a^x) = (\ln a)a^x$$

Note that when $a = e$, then, we have our special natural exponential function and its derivative.

$$\frac{d}{dx}(e^x) = (\ln e)e^x = e^x$$

Example 7.6. Find the derivative of $y = 8^{\sin x}$.

Solution. We use the Chain Rule with the outer function 8^x and the inner function $\sin x$ since their composition gives f. The derivative of 8^x at $\sin x$ is $(\ln 8)8^{\sin x}$ and the derivative of $\sin x$ is $\cos x$. Hence, by the Chain Rule,

$$f'(x) = (\ln 8)8^{\sin x} \cos x$$

□

7.2.3. Derivative of the Logarithmic Function

The general logarithmic function is of the form

$$f(x) = \log_a x, \quad a > 0, x > 0, a \neq 1$$

When $a = e$ we have the natural logarithmic function which we denote by $\ln x$. The derivative of the natural logarithmic function is

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

So what is the derivative of the general logarithmic function? By the change of bases formula for logarithms,

$$\log_a x = \frac{\log_b x}{\log_b a}$$

If we pick $b = e$, then

$$\log_a x = \frac{\log_e x}{\log_e a} = \frac{\ln x}{\ln a}$$

Therefore,

$$\frac{d}{dx}(\log_a x) = \frac{d}{dx}\left(\frac{\ln x}{\ln a}\right) = \frac{1}{x \ln a}$$

Example 7.7. Find the equation of the tangent line to the curve $y = \ln(\sin x)$ at $x = \frac{\pi}{4}$.

Solution. To find the equation of the tangent line we first have to find the slope which amounts to finding the derivative of the function at $x = \frac{\pi}{4}$

We find the derivative of $\ln(\sin x)$ by using the Chain Rule. The outer function in this case is $\ln x$ and the inner function is $\sin x$ since their composition gives $\ln(\sin x)$. The derivative of the outer function at the inner function is $\frac{1}{\sin x} = \csc x$ and the derivative of $\sin x$ is $\cos x$. Hence, by the Chain Rule,

$$\frac{dy}{dx} = \left(\frac{1}{\sin x}\right) \cos x = \cot x$$

and

$$\left.\frac{dy}{dx}\right|_{x=\frac{\pi}{4}} = \cot(\frac{\pi}{4}) = 1$$

Now, using the point slope form we have,

$$\begin{aligned} y - f(\frac{\pi}{4}) &= 1(x - \frac{\pi}{4}) \\ y - \ln(\frac{1}{\sqrt{2}}) &= x - \frac{\pi}{4} \\ y &= x - \frac{\pi}{4} + \ln(\frac{1}{\sqrt{2}}) \end{aligned}$$

□

Example 7.8. Find the derivative of $y = \log_3(x^2 - 1)$.

Solution. We use the change of base formula to write

$$y = \log_3(x^2 - 1) = \frac{\ln(x^2 - 1)}{\ln 3}$$

Therefore,

$$\frac{dy}{dx} = \frac{1}{\ln 3} \cdot \frac{d}{dx}(\ln(x^2 - 1))$$

By the Chain Rule,

$$\frac{d}{dx}(\ln(x^2 - 1)) = \frac{2x}{x^2 - 1}$$

Hence,

$$\frac{dy}{dx} = \frac{2x}{(\ln 3)(x^2 - 1)}$$

□

7.2.4. Logarithmic Differentiation

Sometimes certain functions are easy to differentiate if we use the rules of logarithm first on them. Let's look at the function

$$y = \frac{(x^3 - 2)(2x^2 - 3x + 5)^3}{(4x^3 - 1)^2}$$

If we want to find $\frac{dy}{dx}$ then we have to use the quotient rule and then the product rule and also the chain rule at some point. The whole operation would be messy and would take a long time. Instead, we use the rules to logarithm to write the expression as a sum and difference of logarithms. We first apply natural logarithm (logarithm with any base can be similarly applied) on both sides.

$$\begin{aligned}
\ln y &= \ln \frac{(x^3 - 2)(2x^2 - 3x + 5)^3}{(4x^3 - 1)^2} \\
\ln y &= \ln(x^3 - 2) + \ln(2x^2 - 3x + 5)^3 - \ln(4x^3 - 1)^2 \\
\ln y &= \ln(x^3 - 2) + 3\ln(2x^2 - 3x + 5) - 2\ln(4x^3 - 1)
\end{aligned}$$

Now we apply the derivative with respect to x on both sides,

$$\frac{d}{dx}(\ln y) = \frac{d}{dx}(\ln(x^3 - 2)) + \frac{d}{dx}(3\ln(2x^2 - 3x + 5)) - \frac{d}{dx}(2\ln(4x^3 - 1))$$

By the Chain Rule, the derivative of $\ln y$ with respect to x is $\frac{1}{y} \cdot \frac{dy}{dx}$. This is because, we can pick our outer function to be $\ln x$ and our inner function to be y, since their composition gives $\ln y$. Hence, the derivative of the outer function at the inner function is $\frac{1}{y}$ and the derivative of the inner function y with respect to x is just $\frac{dy}{dx}$. Therefore applying the Chain Rule to each of the functions on the right we have,

$$\begin{aligned}\frac{1}{y}\cdot\frac{dy}{dx} &= \frac{3x^2}{x^3-2}+\frac{3(4x-3)}{2x^2-3x+5}-\frac{24x^2}{4x^3-1}\\ \frac{dy}{dx} &= y\Big(\frac{3x^2}{x^3-2}+\frac{3(4x-3)}{2x^2-3x+5}-\frac{24x^2}{4x^3-1}\Big)\end{aligned}$$

Replacing the y by the original equation we have,

$$\frac{dy}{dx}=\frac{(x^3-2)(2x^2-3x+5)^3}{(4x^3-1)^2}\Big(\frac{3x^2}{x^3-2}+\frac{3(4x-3)}{2x^2-3x+5}-\frac{24x^2}{4x^3-1}\Big)$$

□

Let us look at another example.

Example 7.9. Find the derivative of $y = x^{e^x}$.

Solution. Here we apply the natural logarithm again to make the function simpler to differentiate.

$$\begin{aligned}\ln y &= \ln x^{e^x}\\ \ln y &= e^x \ln x\end{aligned}$$

Now we apply the derivative with respect to x on both sides and as before we use the Chain Rule and Product Rule to find the derivative of the functions.

$$\begin{aligned}\frac{d}{dx}(\ln y) &= \frac{d}{dx}(e^x\ln x)\\ \frac{1}{y}\cdot\frac{dy}{dx} &= \frac{e^x}{x}+e^x\ln x\\ \frac{dy}{dx} &= y\Big(\frac{e^x}{x}+e^x\ln x\Big)=x^{e^x}\left(\frac{e^x}{x}+e^x\ln x\right)\end{aligned}$$

□

Handout 8
MATH 251

Dibyajyoti Deb

8.1. Implicit Differentiation

8.1.1. Performance Criteria

(a) Use the rules of differentiation to find derivatives explicitly and implicitly.
(b) Use implicit differentiation to calculate implicit derivatives of inverse trigonometric functions and implicit equations.

8.1.2. Implicit Functions

Until now we have seen and worked with "explicit" functions. These are functions where the y and x are separated i.e. if y is a function of x then,

$$y = f(x)$$

Now, we look at "implicit" functions. These are functions where the x and the y are not separated. Hence these are of the form

$$f(x, y) = 0$$

An example of a function of this type is

$$xy^2 + x^2y = x^3 + 3$$

In this section we look at methods by which we can find dy/dx for an implicit function of x and y.

8.1.3. Implicit Differentiation

To find dy/dx of an implicit function of x and y, we follow few simple steps. It is best if we look at a specific example.

Example 8.1. Find dy/dx for

$$x^2 + y^2 = 4$$

Solution. We follow these steps to find the answer.

Step 1. We take the derivative with respect to x on both sides. Hence, we have,

$$\begin{aligned}\frac{d}{dx}(x^2+y^2) &= \frac{d}{dx}(4)\\ \frac{d}{dx}(x^2)+\frac{d}{dx}(y^2) &= \frac{d}{dx}(4)\end{aligned}$$

Step 2. Now we find the derivatives of each of these functions separately. The derivative of x^2 with respect to x is $2x$. To find the derivative of y^2 with respect to x we use the Chain Rule. The outer function is x^2 and the inner function is y, since their composition gives y^2.

By the Chain Rule we find the derivative of the outer function with respect to x to get $2x$. When we evaluate this result at the inner function which is y we get $2y$. The derivative of the inner function y, with respect to x is dy/dx. Hence, by the Chain Rule,

$$\frac{d}{dx}(y^2) = 2y\frac{dy}{dx}$$

The derivative of 4 with respect to x is 0, since 4 is a constant. Hence, we have

$$2x + 2y\frac{dy}{dx} = 0$$

Step 3. Once we have found the derivative on both sides with respect to x, we then solve for dy/dx.

$$\begin{aligned}2x+2y\frac{dy}{dx} &= 0\\ 2y\frac{dy}{dx} &= -2x\\ \frac{dy}{dx} &= -\frac{x}{y}\end{aligned}$$

□

We look at some more examples.

Example 8.2. Using implicit differentiation find dy/dx for

$$\sin(x+y) = x + \cos y$$

Solution. We follow these steps to find the answer.

Step 1. We take the derivative with respect to x on both sides. Hence, we have,

$$\begin{aligned}\frac{d}{dx}(\sin(x+y)) &= \frac{d}{dx}(x+\cos y)\\ \frac{d}{dx}(\sin(x+y)) &= \frac{d}{dx}(x)+\frac{d}{dx}(\cos y)\end{aligned}$$

Step 2. We now find the derivative of each of these functions separately. To find the derivative of $\sin(x+y)$, we use the Chain Rule. In this case, the outer function is $\sin x$ and the inner function is $x+y$, since their composition is $\sin(x+y)$.

By the Chain Rule, the derivative of $\sin x$ with respect to x is $\cos x$. When we evaluate this result at the inner function $x+y$ we have $\cos(x+y)$. The derivative of the inner function $x+y$ with respect to x is

$$\frac{d}{dx}(x+y)=\frac{d}{dx}(x)+\frac{d}{dx}(y)=1+\frac{dy}{dx}$$

Hence,

$$\frac{d}{dx}(\sin(x+y))=\cos(x+y)\left(1+\frac{dy}{dx}\right)$$

The derivative of x with respect to x is 1. To find the derivative of $\cos y$ with respect to x, we use the Chain Rule. In this case, the outer function is $\cos x$ and the inner function is y, since their composition is $\cos y$. The derivative of the outer function is $-\sin x$ which when evaluated at the inner function y, gives $-\sin y$. The derivative of the inner function y with respect to x is dy/dx. Hence,

$$\frac{d}{dx}(\cos y)=-\sin y\frac{dy}{dx}$$

Putting them together we have,

$$\begin{aligned}\frac{d}{dx}(\sin(x+y)) &= \frac{d}{dx}(x)+\frac{d}{dx}(\cos y)\\ \cos(x+y)\left(1+\frac{dy}{dx}\right) &= 1-\sin y\frac{dy}{dx}\end{aligned}$$

Step 3. We now solve for dy/dx.

$$\begin{aligned}
\cos(x+y)\Big(1+\frac{dy}{dx}\Big) &= 1-\sin y\frac{dy}{dx}\\
\cos(x+y)+\cos(x+y)\frac{dy}{dx} &= 1-\sin y\frac{dy}{dx}\\
\cos(x+y)\frac{dy}{dx}+\sin y\frac{dy}{dx} &= 1-\cos(x+y)\\
\frac{dy}{dx}\Big(\cos(x+y)+\sin y\Big) &= 1-\cos(x+y)\\
\frac{dy}{dx} &= \frac{1-\cos(x+y)}{\cos(x+y)+\sin y}
\end{aligned}$$

□

Example 8.3. Find the equation of the tangent line at the point $(2,1)$ to the curve

$$xy+x^2y^2=5$$

Solution. To find the equation of the tangent line we first have to find dy/dx at the specified point. To find dy/dx, we follow these steps.

Step 1. We take the derivative with respect to x on both sides. Hence, we have,

$$\begin{aligned}
\frac{d}{dx}(xy+x^2y^2) &= \frac{d}{dx}(5)\\
\frac{d}{dx}(xy)+\frac{d}{dx}(x^2y^2) &= \frac{d}{dx}(5)
\end{aligned}$$

Step 2. We now find the individual derivatives. To find the derivative of xy with respect to x, we use the Product rule,

$$\begin{aligned}
\frac{d}{dx}(xy) &= x\frac{d}{dx}(y)+y\frac{d}{dx}(x)\\
&= x\frac{dy}{dx}+y
\end{aligned}$$

To find the derivative of x^2y^2 with respect to x, we again use the Product rule since x^2y^2 is the product of two functions x^2 and y^2,

$$\begin{aligned}
\frac{d}{dx}(x^2y^2) &= x^2\frac{d}{dx}(y^2)+y^2\frac{d}{dx}(x^2)\\
&= x^2\cdot 2y\frac{dy}{dx}+y^2\cdot 2x\\
&= 2x^2y\frac{dy}{dx}+2xy^2
\end{aligned}$$

Since 5 is a constant, therefore

$$\frac{d}{dx}(5) = 0$$

Putting them all together we have

$$\begin{aligned} \frac{d}{dx}(xy) + \frac{d}{dx}(x^2y^2) &= \frac{d}{dx}(5) \\ x\frac{dy}{dx} + y + 2x^2y\frac{dy}{dx} + 2xy^2 &= 0 \end{aligned}$$

Step 3. We now solve for dy/dx.

$$\begin{aligned} x\frac{dy}{dx} + y + 2x^2y\frac{dy}{dx} + 2xy^2 &= 0 \\ x\frac{dy}{dx} + 2x^2y\frac{dy}{dx} &= -y - 2xy^2 \\ \frac{dy}{dx}(x + 2x^2y) &= -y - 2xy^2 \\ \frac{dy}{dx} &= \frac{-y - 2xy^2}{x + 2x^2y} \end{aligned}$$

We find the slope of the tangent line at $(2, 1)$ by finding

$$\left.\frac{dy}{dx}\right|_{(2,1)} = \frac{-1 - 2 \cdot 2 \cdot 1^2}{2 + 2 \cdot 2^2 \cdot 1} = -\frac{1}{2}$$

We use the point-slope form to find the equation of the tangent line.

$$\begin{aligned} y - 1 &= -\frac{1}{2}(x - 2) \\ y - 1 &= -\frac{x}{2} + 1 \\ y &= -\frac{x}{2} + 2 \end{aligned}$$

□

8.2. Related Rates

8.2.1. Performance Criteria

(a) Set up and solve word-problems with related differential rates of change.

8.2.2. Related Rates

Now, we look at applications of the derivative. We have seen that the derivative of y with respect to x (i.e. dy/dx) computes the rate of change of y with respect to x. In this section we calculate the unknown rate of change in terms of other rates of change that are known.

We follow 3 steps in order to solve problems in this section.

Step 1. As the initial problem is written in sentences, therefore, we assign variables and restate the problem. It is sometimes useful if we draw a picture describing the problem.

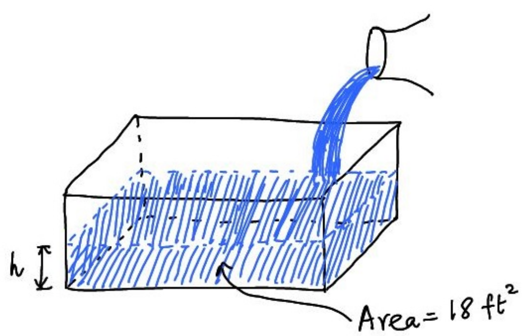
Step 2. We next find an equation relating the variables and then differentiate to introduce the derivatives.

Step 3. Now we use the information provided to find the unknown derivative.

Let us use these steps to solve few problems. Remember, every problem is different and there is no specific way of solving a problem. We might have to use results in elementary geometry or trigonometry to construct the equation that we need to differentiate.

Example 8.4. Consider a rectangular bathtub whose base is 18 ft^2. At what rate is water pouring into the tub if the water level rises at a rate of 0.8 ft/min?

Solution. We begin by drawing a picture of the bathtub. Let us denote the height of water at any instant of time t by h.



Step 1. Since we have to find the rate at which water is being poured, therefore if we denote the volume of water in the tub at any instant of time t by $V(t)$, then we are asked to find dV/dt.

Now, since the height (h) of the water in the tub rises at the rate of 0.8 ft/min, therefore $dh/dt = 0.8$. Hence,

$$\text{Find } \frac{dV}{dt} \quad \text{given that} \quad \frac{dh}{dt} = 0.8$$

Step 2. Now we find an equation relating the variables and then differentiate both sides to introduce the derivatives. Since, the height of the water level at any instant of time t is h, hence the volume of the water at that instant is

$$V(t) = \text{Area of the base } \times h = 18h$$

We take the derivative with respect to t on both sides to get,

$$\frac{dV}{dt} = 18\frac{dh}{dt}$$

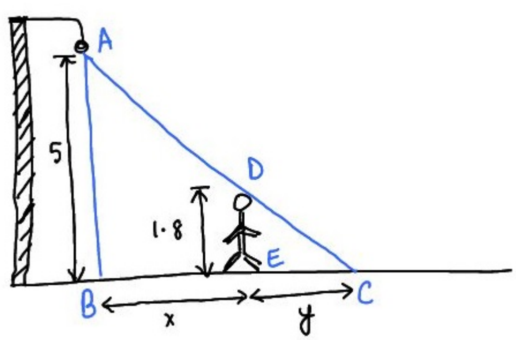
Step 3. Now we find dV/dt given that $dh/dt = 0.8$. Hence,

$$\frac{dV}{dt} = 18 \times 0.8 = 14.4 \text{ ft}^3/\text{min}.$$

□

Example 8.5. A man of height 1.8 meters walks away from a 5-meter lamppost at a speed of 1.2 m/s. Find the rate at which his shadow is increasing in length.

Solution. We first draw a picture of this problem.



Step 1. We denote the distance of man from the lamppost at any instant by x and the length of his shadow at the same instant by y. Since the rate at which the man is walking away from the lamppost is 1.2 m/s therefore, $dx/dt = 1.2$. We have to find the rate at which his shadow is increasing in length, i.e. we are asked to find dy/dt. Hence,

$$\text{Find } \frac{dy}{dt} \quad \text{given that} \quad \frac{dx}{dt} = 1.2$$

Step 2. Now we find an equation involving the variables x and y. To find an equation involving x and y, we notice that we have two similar triangles in the picture. The triangle ABC is similar to the triangle DEC. Comparing the ratio of their sides we have

$$\begin{aligned}
\frac{AB}{DE} &= \frac{BC}{EC} \\
\frac{5}{1.8} &= \frac{x+y}{y} \\
5y &= 1.8x + 1.8y \\
3.2y &= 1.8x
\end{aligned}$$

Taking derivatives on both sides with respect to t we have,

$$3.8\frac{dy}{dt} = 1.2\frac{dx}{dt}$$

Step 3. Now we are given $dx/dt = 1.2$ and we are supposed to find dy/dt. Therefore,

$$\begin{aligned}
3.2\frac{dy}{dt} &= 1.8 \times 1.2 = 2.16 \\
\frac{dy}{dt} &= \frac{2.16}{3.2} = 0.675 \text{ m/s}
\end{aligned}$$

Therefore the shadow of the man is increasing at the rate of 0.675 m/s.

□

Handout 9
MATH 251

Dibyajyoti Deb

9.1. Extreme Values

9.1.1. Performance Criteria

(a) Calculate the local maxima and minima and also the absolute max and min of a function on an interval.
(b) Use the derivative of a function to determine where it is increasing and where it is decreasing.
(c) Distinguish between the extrema of a function and the locations of those extrema.

9.1.2. Extrema

In this section we look at how we can find the **extreme values** or **extrema** of a function. In other words we will look at how we can find the maximum and minimum of a function f on an interval I.

There are two types of extrema that we will be discussing. **Global** or **Absolute extrema** and **Local extrema**. We first look at the definition of **local extrema**.

Definition 9.1. Local Extrema - We say that $f(x)$ has a

- **Local minimum** at $x = a$ if $f(a)$ is the minimum value of f on some open interval (in the domain of f) containing a.
- **Local maximum** at $x = a$ if $f(a)$ is the maximum value of f on some open interval (in the domain of f) containing a.

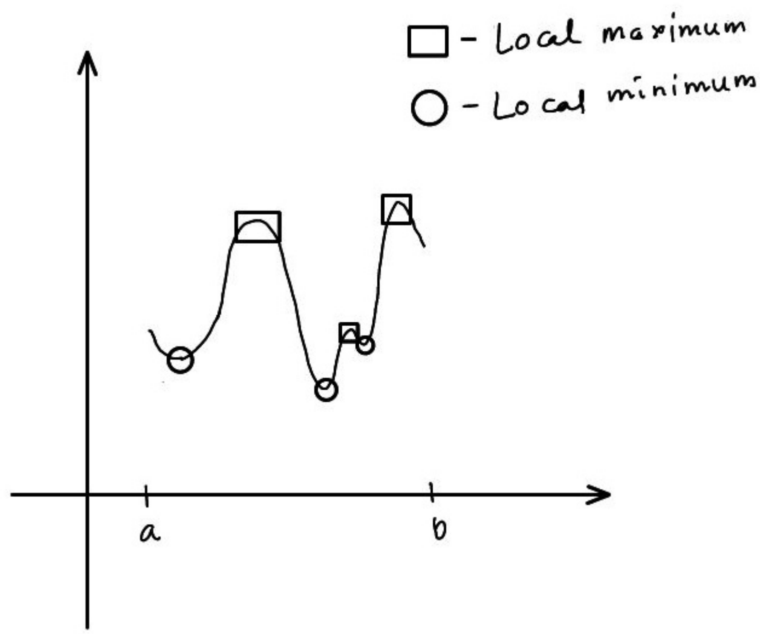
This means that if a local maximum occurs at $x = a$, then the point $(a, f(a))$ is the highest point (peak) within some neighborhood of $(a, f(a))$ on the graph. Similarly if a local minimum occurs at $x = a$, then the point $(a, f(a))$ is the lowest point (valley) within some neighborhood of $(a, f(a))$ on the graph.

Now let's look at the definition of **absolute extrema**.

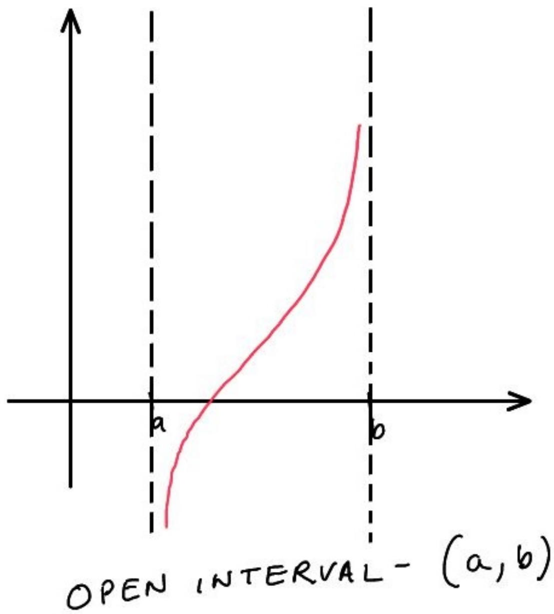
Definition 9.2. Absolute Extrema - We say that $f(x)$ has an

- **Absolute minimum** at $x = a$ on an interval I ($a \in I$) if $f(a) \leq f(x)$ for all $x \in I$.
- **Absolute maximum** at $x = a$ on an interval I ($a \in I$) if $f(a) \geq f(x)$ for all $x \in I$.

This means that to find the absolute maximum, we look for the largest maximum (highest peak) among all the local maximum's. Similarly, to find the absolute minimum, we looks for the smallest minimum (lowest valley) among all the local minimum's. The next picture shows absolute extrema and local extrema in one graph.



What happens if the interval I is an open interval? We can see cases where their might not be any extrema (local or global), i.e. their may not be any peaks or valleys. This is shown in the next picture.



However, we don't have this problem when we have a closed interval I. Thus,

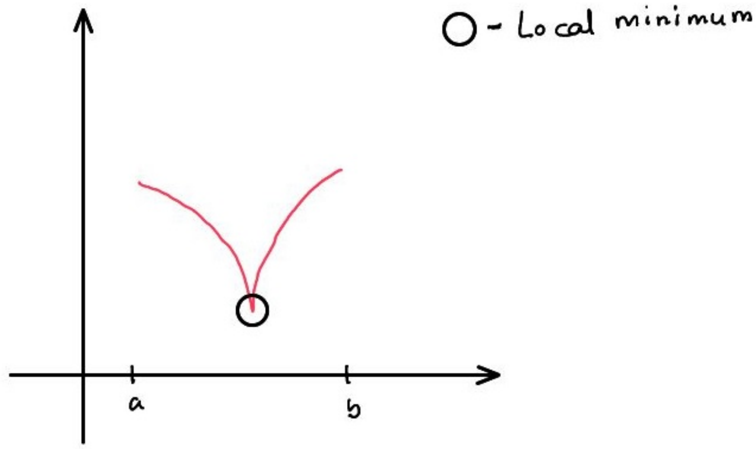
Theorem 9.3. *A continuous function $f(x)$ on a closed (bounded) interval $I = [a, b]$ takes on both a minimum and a maximum value on I.*

9.1.3. Critical Points

Now, we ask ourselves the question,

Given a function $f(x)$, how do we find its local extrema?

If we look at the pictures above, we notice that the tangent line is horizontal at points where there is a local maximum or local minimum. Is this sufficient enough to detect peaks (local maximum) and valleys (local minimum)? What if we have a graph like the one below.



We see above that the local minimum occurs at a point where the function is not differentiable (since it has a corner at that point). The tangent line at that point is not defined. Thus,

> **Definition 9.4. Critical Points** - A point a in the domain of f is called a **critical point** if either $f'(a) = 0$ or $f'(a)$ does not exist.

Example 9.5. Find the critical points of

$$f(x) = \sin^{-1} x - 2x$$

Solution. The domain of f is the interval $[-1, 1]$. Thus, we want to find points on this interval where $f'(x)$ is either equal to zero or is undefined. Now,

$$f'(x) = \frac{1}{\sqrt{1-x^2}} - 2$$

$$\begin{aligned}
f'(x) &= 0 \\
\frac{1}{\sqrt{1-x^2}} - 2 &= 0 \\
\sqrt{1-x^2} &= \frac{1}{2} \\
1 - x^2 &= \frac{1}{4} \\
x^2 &= \frac{3}{4} \\
x &= \pm\frac{\sqrt{3}}{2}
\end{aligned}$$

Now $f'(x)$ is not defined at $x = \pm 1$ (that is where $x^2 - 1 = 0$). Hence, the critical points of f are

$$\pm\frac{\sqrt{3}}{2}, \pm 1$$

□

We could now ask ourselves the question,

Does every critical point a make $f(a)$ a local extrema?

This is not true as if we take the function $f(x) = x^3$, we see that $f'(x) = 3x^2$ which makes $x = 0$ a critical point, however, $f'(0)$ is neither a local maximum nor a local minimum. Thus,

Theorem 9.6. *If $f(a)$ is a local minimum of maximum, then a is a critical point of $f(x)$ but not the other way around.*

We have now arrived at a point where we need to know how to find the extreme values. In the next subsection we look at how we can find the extreme values on a closed interval.

9.1.4. Extreme Values on a Closed Interval

From previous excursions we have seen that if $f(a)$ is a local extremum then a is a critical point. Since the interval is closed, therefore, there is a possibility for the extremum to happen at the endpoints of the interval. Hence we will follow these steps to find the local extremum on a closed interval $[a, b]$.

Step 1. Find the critical points on the interval $[a, b]$.

Step 2. Find the values $f(a), f(b)$ at the endpoints and $f(c)$ for every critical points c from Step 1.

Step 3. The maximum of all values from Step 2 is the absolute maximum of $f(x)$ on $[a, b]$ and the minimum of all values from Step 2 is the absolute minimum of $f(x)$ on $[a, b]$.

Let's look at an example where we use this strategy.

Example 9.7. Find the maximum and minimum of the function on the given interval.

$$f(x) = x^3 - 24\ln x, \qquad \left[\frac{1}{2}, 3\right]$$

Solution. Step 1. We first find the critical points of f.

$$f'(x) = 3x^2 - \frac{24}{x}$$

$$\begin{aligned} f'(x) &= 0 \\ 3x^2 - \frac{24}{x} &= 0 \\ 3x^3 &= 24 \\ x^3 &= 8 \\ x &= 2 \end{aligned}$$

$f'(x)$ is not defined at $x = 0$, however, 0 does not lie on the interval $[\frac{1}{2}, 3]$. Hence, the only critical point of $f(x)$ is at $x = 2$.

Step 2. We find,

$$f\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^3 - 24\ln\left(\frac{1}{2}\right) = 16.76$$

$$f(3) = 3^3 - 24\ln(3) = 0.63$$

$$f(2) = 2^3 - 24\ln(2) = -8.64$$

Step 3. Hence the maximum of $f(x)$ on $[\frac{1}{2}, 3]$ is 16.76 which is attained at $x = \frac{1}{2}$ and the minimum is -8.64 which is attained at $x = 2$.

□

9.2. L'Hôpital's Rule

9.2.1. Performance Criteria

(a) Calculate limits with LHôpital's rule.

9.2.2. Indeterminate Forms

We have looked at limits earlier. In this section we look at special types of functions which become indeterminate at the point where we have to evaluate the limit. Consider the limit

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

If $f(a) = 0$ and $g(a) = 0$ then we say that $f(x)/g(x)$ has an *indeterminate form*. This is also true if $f(a) \to \infty$ and $g(a) \to \infty$. In this case how do we evaluate the limit of the quotient? The next theorem answers this question.

Theorem 9.8. L'Hôpital's Rule *Assume that $f(x)$ and $g(x)$ are differentiable on an open interval containing a and that*

$$f(a) = g(a) = 0$$

Also assume that $g'(x) \neq 0$ (except possibly at a). Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

if the limit on the right exits or is infinite. This conclusion also holds if $f(x)$ and $g(x)$ are differentiable for x near a and

$$\lim_{x \to a} f(x) = \pm\infty \qquad \text{and} \ \lim_{x \to a} g(x) = \pm\infty$$

What the above theorem states is that if $f(x)/g(x)$ is of the form $0/0$ or ∞/∞ as x approaches a, then we can instead find the limit of $f'(x)/g'(x)$ as approaches a and so on.

Example 9.9. The form ∞/∞.

Evaluate the limit.

$$\lim_{x \to \infty} \frac{3x^3 + 2x^2}{5x^3 - 4}$$

Solution. Since $3x^3 + 2x^2 \to \infty$ as $x \to \infty$ and $5x^3 - 4 \to \infty$ as $x \to \infty$, hence the quotient is the indeterminate form ∞/∞. We apply the L'Hôpital's Rule,

$$\lim_{x \to \infty} \frac{3x^3 + 2x^2}{5x^3 - 4} = \lim_{x \to \infty} \frac{9x^2 + 4x}{15x^2}$$

Now we evaluate this new limit. We see that $9x^2 + 4x \to \infty$ as $x \to \infty$ and $15x^2 \to \infty$ as $x \to \infty$, hence the new quotient is also an indeterminate form ∞/∞. Therefore, we still apply the L'Hôpital's Rule,

$$\lim_{x \to \infty} \frac{9x^2 + 4x}{15x^2} = \lim_{x \to \infty} \frac{18x + 4}{30x}$$

In this new limit we see that $18x + 4 \to \infty$ as $x \to \infty$ and $30x \to \infty$ as $x \to \infty$, hence the new quotient is also an indeterminate form ∞/∞. Therefore, we still keep applying the L'Hôpital's Rule,

$$\lim_{x \to \infty} \frac{18x + 4}{30x} = \lim_{x \to \infty} \frac{18}{30} = \frac{3}{5}$$

□

Let's look at another example.

Example 9.10. The form $0/0$.

Evaluate the limit.

$$\lim_{x \to 0} \frac{e^{2x} - 1 - x}{x^2}$$

Solution. Since $e^{2x} - 1 - x = 0$ when $x = 0$ and $x^2 = 0$ when $x = 0$, therefore the quotient is an indeterminate form $0/0$. We apply the L'Hôpital's Rule,

$$\lim_{x \to 0} \frac{e^{2x} - 1 - x}{x^2} = \lim_{x \to 0} \frac{2e^{2x} - 1}{2x}$$

Now as $2e^{2x} - 1 = 1$ when $x = 0$ and $2x = 0$ when $x = 0$. Thus, this is not an indeterminate form, however the numerator is a finite number while the denominator approaches zero as x approaches zero. Hence, the limit does not exist.

□

What happens if the function is not a quotient? Let's look at an example where we tackle this situation.

Example 9.11. The form $0 \cdot \infty$.

Evaluate the limit.

$$\lim_{x \to 0^+} \sqrt{x} \ln x$$

Solution. For this function as $x \to 0^+$,

- $\sqrt{x} \to 0$
- $\ln x \to -\infty$

Hence, $\sqrt{x} \ln x$ is of the form $0 \cdot \infty$ (We can leave the $-$ sign since it can always be pulled out). This is not an indeterminate form. So we have to somehow write it in the form $0/0$ or ∞/∞ before we can use the L'Hôpital's Rule. We do this by writing the function as

$$\sqrt{x} \ln x = \frac{\ln x}{\frac{1}{\sqrt{x}}}$$

Now as $x \to 0^+$,

- $\ln x \to -\infty$
- $1/\sqrt{x} \to \infty$

Hence, this new form of the function is of the form $-\infty/\infty$ so we apply L'Hôpital's Rule to get,

$$\begin{aligned} \lim_{x \to 0^+} \frac{\ln x}{\frac{1}{\sqrt{x}}} &= \lim_{x \to 0^+} \frac{\frac{1}{x}}{-\frac{1}{2}x^{-3/2}} \\ &= \lim_{x \to 0^+} \frac{-2x^{3/2}}{x} \\ &= \lim_{x \to 0^+} -2x^{1/2} = 0. \end{aligned}$$

□

Let's look at another function that is not a quotient but a difference.

Example 9.12. The form $\infty - \infty$.

Evaluate the limit.

$$\lim_{x\to 0}\left(\cot x - \frac{1}{x}\right)$$

Solution. Here are $x \to 0$,

- $\cot x \to \infty$
- $1/x \to \infty$

Hence, this function is of the form $\infty - \infty$, which is an indeterminate form. We have to now write it in the form $0/0$ or ∞/∞ before we can use the L'Hôpital's Rule. We do this by combining the functions using a common denominator.

$$\cot x - \frac{1}{x} = \frac{\cos x}{\sin x} - \frac{1}{x} = \frac{x\cos x - \sin x}{x \sin x}$$

This new form of the function is of the form $0/0$ so we apply L'Hôpital's Rule to get,

$$\begin{aligned}
\lim_{x\to 0}\left(\frac{x\cos x - \sin x}{x\sin x}\right) &= \lim_{x\to 0}\frac{-x\sin x + \cos x - \cos x}{x\cos x + \sin x} \\
&= \lim_{x\to 0}\frac{-x\sin x}{x\cos x + \sin x} \\
&= \lim_{x\to 0}\frac{-x\cos x - \sin x}{-x\sin x + \cos x + \cos x} \\
&= \lim_{x\to 0}\frac{-x\cos x - \sin x}{-x\sin x + 2\cos x}
\end{aligned}$$

The last limit can be evaluated by substituting 0 for x to get

$$\lim_{x\to 0}\frac{-x\cos x - \sin x}{-x\sin x + 2\cos x} = \frac{-0-0}{-0+2} = 0$$

□

Now let's look at another form.

Example 9.13. The form 0^0.

Evaluate the limit.

$$\lim_{x\to 0^+} x^x$$

Solution. Here as $x \to 0^+$,

- $x \to 0$
- $\sin x \to 0$

Hence, this function is of the form 0^0. We compute the limit of the logarithm $\ln x^x = x \ln x$.

We see that $x \ln x$ is of the form $0 \cdot \infty$ as $x \to 0^+$. Hence,

$$\lim_{x\to 0^+} \ln x^x = \lim_{x\to 0^+} x \ln x = \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}} = \lim_{x\to 0^+} \frac{\frac{1}{x}}{-x^{-2}} = \lim_{x\to 0^+} (-x) = 0$$

By property of limits,

$$\lim_{x\to 0^+} \ln(x^x) = \ln(\lim_{x\to 0^+} x^x) = 0$$

Exponentiating both sides we get,

$$\lim_{x\to 0^+} x^x = e^0 = 1.$$

□