These are lecture notes that I typed myself for the Spring 2012 class that I taught titled “Elementary Differential Equations”.

These lecture notes were used by the students in their preparation for the exams.

An Introduction to Ordinary Differential Equations Lecture 1

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1.1. Outline of Lecture

- What is a Differential Equation?
- Solutions of Some Differential Equations
- Classifying Diff. Eqns.: Order, Linear vs. Nonlinear

1.2. What is a Differential Equation?

Many of the principles, or laws, underlying the behavior of the natural world are statements or relations involving rates at which things happen. When expressed in mathematical terms, the relations are equations and the rates are derivatives. Equations containing derivatives are **differential equations**. You've probably all seen an ordinary differential equation (ODE); for example the physical law that governs the motion of objects is Newton's second law, which states that the mass of the object times its acceleration is equal to the net force on the object. In mathematical terms this law is expressed by the equation

$$F = m\frac{dv}{dt}. \tag{1.1}$$

where m is the mass of the object, v is its velocity, t is the time and F is the net force exerted on the object. Here t is the **independent variable** and v is the **dependent variable**. This is an ODE because there is only one independent variable, here t which represents time.

A partial differential equation (PDE) relates the partial derivatives of a function of two or more independent variables together. For example, the heat conduction equation,

$$\alpha^2\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial u(x,t)}{\partial t} \tag{1.2}$$

arises in many places in mathematics and physics. Here α^2 is a physical constant. For simplicity, we can use subscript notation for partial derivatives, so this equation can also be written $\alpha^2 u_{xx} = u_t$.

We say a function is a **solution** to a differential equation if it satisfies the equation and any side conditions given. Mathematicians are often interested in if a solution **exists** and when it is **unique**.

1.3. Solution to some differential equations

Let's look into the differential equation from the previous section in more detail. What are the forces that act on the object as it falls?

Gravity exerts a force equal to the weight of the object, or mg, where g is the acceleration due to gravity. There is also a force due to air resistance, or drag, that is more difficult to model. Let's assume that the drag force is proportional to the velocity. Thus the drag force has a magnitude γv, where γ is a constant called the drag coefficient.

Gravity always acts in the downward (positive) direction, whereas drag acts in the upward (negative) direction. Thus

$$m\frac{dv}{dt} = mg - \gamma v \tag{1.3}$$

To solve this equation, divide by m. Note that m, g and γ are constants for this model.

$$\frac{dv}{dt} = g - \frac{\gamma}{m}v \tag{1.4}$$

We would like to isolate the terms involving v and t such that we can integrate both sides.

$$\frac{dv}{dt} = -\frac{\gamma}{m}(v - \frac{gm}{\gamma}) \tag{1.5}$$

After cross multiplying

$$\frac{dv}{v - \frac{gm}{\gamma}} = -\frac{\gamma}{m}dt \tag{1.6}$$

Integrate both sides

$$\int \frac{dv}{v - \frac{gm}{\gamma}} = \int -\frac{\gamma}{m}dt$$

$$\ln|v - \frac{gm}{\gamma}| = -\frac{\gamma t}{m} + C$$

$$v - \frac{gm}{\gamma} = \pm e^{-\frac{\gamma t}{m} + C}$$

Simplifying the right side and replacing $\pm e^C$ with C_1 (another constant), we have,

$$v(t) = \frac{gm}{\gamma} + C_1 e^{-\frac{\gamma t}{m}} \tag{1.7}$$

This expression contains all possible solutions of Eq. (1.4) and is called the **general solution**. The geometrical representation of the general solution 1.7 is an infinite family of curves called **integral curves**. Each integral curve is associated with a particular value of C_1 and is the graph corresponding to that value of C_1.

Now if the ball is "dropped" from a certain height then it is clear that the initial velocity is zero. Therefore $v(0) = 0$. We can use this additional condition to determine C_1. This is an example of an **initial condition**. The differential equation together with the initial condition form an **initial value problem**.

1.4. Classifying Diff. Eqns.: Order, Linear vs. Nonlinear

We have already discussed the difference between an ordinary and a partial differential equation. When classifying differential equations we need to look at the *order* of a differential equation. The **order** of a differential equation is the order of the highest derivative present in the equation. For example Eq. 1.4 is a first order ODE. More generally, the equation

$$F[t, x(t), x'(t), \ldots, x^{(n)}(t)] = 0 \tag{1.8}$$

is an ordinary differential equation of the nth order.

A crucial classification of differential equations is whether they are linear or nonlinear. The ordinary differential equation

$$f(t, y, y', \ldots, y^{(n)}) = 0, \tag{1.9}$$

is said to be **linear** if F is a linear function of the variable $y, y', \ldots, y^{(n)}$; a similar definition applies to partial differential equations. Thus the general linear ordinary differential equation of order n is

$$a_n(t)y^{(n)} + a_{n-1}(t)y^{(n-1)} + \ldots + a_0(t)y = g(t). \tag{1.10}$$

The most important thing to note about linear differential equations is that there are no products of the function, $y(t)$, and its derivatives and neither the function or its derivatives are used in determining if a differential equation is linear. For example Eq. 1.4 is a linear equation.

An equation that is not of the form 1.10 is a **nonlinear** equation. An example of a nonlinear equation would be

$$y'' + 2yy' = t^3 \tag{1.11}$$

because of the presence of the term yy'. We will be looking at differences between linear and nonlinear equations in more detail in a later lecture.

First Order Differential Equations Lecture 2

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2.1. Outline of Lecture

- Linear Equations; Method of Integrating Factors
- Separable Equations
- Modeling with First Order Equations

2.2. Linear Equations; Method of Integrating Factors

The most general first order equation is of the form

$$\frac{dy}{dt} = f(t, y) \tag{2.1}$$

where f is a given function of two variables. Any differential function $y = \phi(t)$ that satisfies this equation for all t in some interval is called a solution, and the object is to determine whether such functions exist and, if so, to develop methods for finding them. Unfortunately, for an arbitrary function f, there is no general method of solving the equation in terms of elementary functions.

If the function f in Eq. (2.1) depends linearly on the dependant variable y, then Eq. (2.1) is called a first order linear equation.

The general **first order linear equation** is of the form

$$\frac{dy}{dt} + p(t)y = g(t), \tag{2.2}$$

where both $p(t)$ and $g(t)$ are continuous functions.

The method described in the previous lecture to solve the differential equation describing the motion of the falling object doesn't work here. So we need a different method of solution for it. It involves multiplying the differential equation (2.2) by a certain function $\mu(t)$, chosen so that the resulting equation is readily integrable. The function $\mu(t)$ is called an **integrating factor**.

We look into the method of finding the integrating factor briefly. We multiply Eq. (2.2) by $\mu(t)$, obtaining

$$\mu(t)\frac{dy}{dt} + \mu(t)p(t)y = \mu(t)g(t) \tag{2.3}$$

We see that the left side of Eq. (2.3) is the derivative of the product $\mu(t)y$ if we assume that $\mu(t)$ will satisfy the following

$$\frac{d\mu(t)}{dt} = p(t)\mu(t) \tag{2.4}$$

We have

$$\frac{d\mu(t)}{\mu(t)} = p(t)\,dt$$

and consequently

$$\ln \mu(t) = \int p(t)\,dt + C.$$

after integrating both sides. By choosing the arbitrary constant C to be zero, we obtain the simplest possible function for μ, namely,

$$\mu(t) = e^{\int p(t)\,dt}. \tag{2.5}$$

An example is shown below which uses the method of integrating factors to solve a first order linear equation.

Example 1. Solve the initial value problem.

$$ty' + 2y = t^2 - t + 1, \qquad y(1) = \frac{1}{2}, t > 0$$

Solution 1. We bring the original equation to the form (2.2) by dividing both sides of the equation by t.

$$y' + \frac{2}{t}y = t - 1 + \frac{1}{t} \tag{2.6}$$

The integrating factor is $\mu(t) = e^{\int \frac{2}{t}\,dt} = t^2$. Multiplying both sides of the above equation by t^2 we get

$$t^2y' + 2ty = t^3 - t^2 + t \tag{2.7}$$

The left side of the equation is the derivative of the product t^2y. Therefore

$$\frac{d(t^2y)}{dt} = t^3 - t^2 + t \tag{2.8}$$

Multiplying both sides by dt and then integrating both sides, we have

$$t^2 y = \frac{t^4}{4} - \frac{t^3}{3} + \frac{t^2}{2} + C \tag{2.9}$$

To find C we use the initial value condition to get

$$\frac{1}{2} = \frac{1}{4} - \frac{1}{3} + \frac{1}{2} + C$$

Therefore $C = \frac{1}{12}$.

Using this value of C in Eq. (2.9), and then dividing by t^2 we have the solution to the initial value problem.

$$y = \frac{t^2}{4} - \frac{t}{3} + \frac{1}{2} + \frac{1}{12t^2} \tag{2.10}$$

2.3. Separable Equations

In this section we look into a subclass of nonlinear equations that can be solved by direct integration. We can rewrite the general first order equation (2.1) in the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{2.11}$$

If it happens that M is a function of x only and N is a function of y only, then the above equation becomes

$$M(x) + N(y)\frac{dy}{dx} = 0 \tag{2.12}$$

Such an equation is said to be **separable**, because it can be written in the differential form

$$M(x)\,dx + N(y)\,dy = 0 \tag{2.13}$$

We can solve the above the equation by integrating the functions M and N. Usually this results in an implicit solution. We illustrate this in the following example.

Example 2. Solve the equation

$$\frac{dy}{dx} = \frac{x^3}{1 - y^2}$$

Solution 2. Cross multiplication makes the equation separable for integration

$$\int (1 - y^2)\,dy = \int x^3\,dx \tag{2.14}$$

Integrating both sides(and bringing the constants on one side) we get

$$y - \frac{y^3}{3} = \frac{x^4}{4} + C$$

Therefore

$$y - \frac{y^3}{3} - \frac{x^4}{4} = C \tag{2.15}$$

where C is an arbitrary constant. Note that the final general solution (2.15) is implicit.

2.4. Modeling with First Order Equations

Now that we have a general idea of first order equations, we can use them to investigate a wide variety of problems in the physical, biological and social sciences. There are three basic steps in solving a problem using differential equation.

- Construction of the model - In this step, translate the physical situation into mathematical terms. The differential equation is a mathematical model of the process.
- Analysis of the model - Once the problem has been formulated mathematically, it's time to solve the one or more differential equations involved with the model (or atleast finding out as much as possible about the properties of the solution).
- Comparison with Experiment or Observation - Finally, having obtained the solution (or atleast some information about it), interpret this information in the context in which the problem arose.

We look at an example below.

Example 3. A tank initially contains 120 L of pure water. A mixture containing a concentration of γ g/L of salt enters the tank at a rate of 3 L/min, and the well-stirred mixture leaves the tank at the same rate. Find an expression in terms of γ for the amount of salt in the tank at any time t. Also find the limiting amount of salt in the tank as $t \to \infty$.

Solution 3. Since the incoming and outgoing flows of water are the same, the amount of water in the tank remains constant at 120 L. Let the amount of salt in the tank at any time t be denoted by $Q(t)$. Thus

$$\frac{dQ}{dt} = \text{rate in - rate out} \tag{2.16}$$

where "rate in" and "rate out" refers to the rate at which the salt flows into and out of the tank, respectively.

$$\text{rate in} = \gamma \text{ g/L} \times 3 \text{ L/min} = 3\gamma \text{ g/min}. \tag{2.17}$$

The concentration of salt in the tank at any time t is $\frac{Q(t)}{120}$ g/L, thus

$$\text{rate out} = \frac{Q(t)}{120} \text{ g/L} \times 3 \text{ L/min} = \frac{Q(t)}{40} \text{ g/min}. \tag{2.18}$$

To make it convenient we omit the units during our calculation. Therefore,

$$\frac{dQ}{dt} = 3\gamma - \frac{Q(t)}{40} \tag{2.19}$$

$$\frac{dQ}{dt} = \frac{120\gamma - Q(t)}{40}$$

This is a first order separable equation (Note that (2.19) is also a first order linear equation).

Cross multiplication makes the equation separable for integration.

$$\int \frac{dQ}{Q(t) - 120\gamma} = \int -\frac{dt}{40} \tag{2.20}$$

Integrating both sides we have,

$$\ln|Q(t) - 120\gamma| = -\frac{t}{40} + C \tag{2.21}$$

To find C, note that the tank initially contains pure water, therefore $Q(0) = 0$. Using this in the above equation we have $C = \ln|120\gamma|$. Putting this value of C back into the above solution and simplifying we have,

$$Q(t) = 120\gamma + |120\gamma|e^{-t/40} \tag{2.22}$$

The initial condition is true if $|120\gamma| = -120\gamma$. Therefore the final solution is

$$Q(t) = 120\gamma - 120\gamma e^{-t/40} \tag{2.23}$$

To find the limiting amount of salt as $t \to \infty$, we find

$$\lim_{t\to\infty} Q(t) = \lim_{t\to\infty}(120\gamma - 120\gamma e^{-t/40}) = 120\gamma.$$

This means that after a very long time the amount of salt in the tank will be 120γ g.

First Order Differential Equations
Lecture 3

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3.1. Outline of Lecture

- Differences Between Linear and Nonlinear Equations
- Exact Equations and Integrating Factors

3.2. Differences between Linear and Nonlinear Equations

We have looked at first order equations so far, both linear and nonlinear. We have developed methods of solving linear equations and some subclasses of nonlinear equations. We now discuss some important ways in which nonlinear equations differ from linear ones.

- **Existence and Uniqueness of Solutions.** So far, we have discusses a number of initial value problems, each of which had a solution and apparently only one solution. This raises the question whether every initial value problem has exactly one solution. The answer to this question is given by the following theorem.

  Theorem 3.1. *If the functions p and g are continuous on an open interval $I : \alpha < t < \beta$ containing the point $t = t_0$, then there exists a unique function $y = \phi(t)$ that satisfies the differential equation*

$$\frac{dy}{dt} + p(t)y = g(t) \tag{3.2}$$

  for each t in I, and that also satisfies the initial condition has a unique solution.

$$y(t_0) = y_0 \tag{3.3}$$

where y_0 is an arbitrary prescribed initial value.

Note that Theorem 3.1 states that the given initial value problem *has* a solution and also that the problem has *only one* solution. In other words, the theorem asserts both the *existence* and *uniqueness* of the solution of the solution of the initial value problem.

We apply this theorem in the next example.

Example 1. Find an interval in which the solution of the initial value problem is certain to exist.

$$(t-3)y' + (\ln t)y = 2t, \qquad y(1) = 2 \tag{3.4}$$

Solution 1. Rewriting the above equation in the standard form, we have

$$y' + \frac{\ln t}{t-3}y = \frac{2t}{t-3}$$

So $p(t) = \dfrac{\ln t}{t-3}$ and $g(t) = \dfrac{2t}{t-3}$. g is continuous for all $t \neq 3$. p is continuous for all $t \neq 0, 3$. Therefore p and g are both continuous on the interval $(-\infty, 0) \cup (0, 3) \cup (3, \infty)$. The interval $(0, 3)$ contains the initial point $t = 1$. Therefore Theorem 3.1 guarantees that the problem has a unique solution on the interval $0 < t < 3$.

We now turn our attention to nonlinear differential equations and modify Theorem 3.1 by a more general theorem.

Theorem 3.5. *Let the function f and $\partial f/\partial y$ be continuous in some rectangle $\alpha < t < \beta, \gamma < t < \delta$ containing the point (t_0, y_0). Then, in some interval $t_0 - h < t < t_0 + h$ containing in $\alpha < t < \beta$, there is a unique solution $y = \phi(t)$ of the initial value problem*

$$y' = f(t, y), \qquad y(t_0) = y_0. \tag{3.6}$$

This is a more general theorem since it reduces to Theorem 3.1 if the differential equation is linear. For then $f(t, y) = -p(t)y + g(t)$ and $\partial f(t, y)/\partial y = -p(t)$, so the continuity of f and $\partial f/\partial y$ is equivalent to the continuity of p and g in this case.

Note that the conditions stated in Theorem 3.5 are sufficient to guarantee the existence of a unique solution of the initial value problem (3.6) in some interval $t_0 - h < t < t_0 + h$, but they are not necessary. That is, the conclusion remains true under slightly weaker hypotheses about the function f. In fact, the existence of solution (but not its uniqueness) can be established on the basis of the continuity of f alone.

We look at an example below making use of the above theorem.

Example 2. Solve the given initial value problem and determine how the interval in which the solution exists depends on the initial value y_0.

$$y' = -\frac{4t}{y}, \qquad y(0) = y_0 \tag{3.7}$$

Solution 2. For this equation $f = -4t/y$ and $\partial f/\partial y = 4t/y^2$. f and $\partial f/\partial y$ are continuous in any rectangle where $y \neq 0$. This is also a separable equation. Cross multiplication makes the equation separable for integration.

$$\int y\, dy = \int -4t\, dt$$

Integrating both sides we have,

$$\frac{y^2}{2} = -2t^2 + C \tag{3.8}$$

for some constant C. Using the initial values we have $C = y_0^2/2$. Using this value for C and simplifying both sides we have

$$y = \pm\sqrt{-4t^2 + y_0^2}$$

However we would like to find out the interval where the solution exists. The term inside the radical has to be non-negative. Therefore $4t^2 < y_0^2$ or $|t| < |y_0|/2$. By Theorem 3.5, we get the extra condition that $y_0 \neq 0$ (Since f and $\partial f/\partial y$ are continuous in any rectangle where $y \neq 0$ containing the point $(0, y_0)$).

Therefore the solution looks like

$$y = \pm\sqrt{-4t^2 + y_0^2}, \quad y_0 \neq 0, \quad |t| < |y_0|/2. \tag{3.9}$$

Now let's look at some other differences between linear and nonlinear equations.

- **Interval of Definition.** According to Theorem 3.1, the solution to a linear equation (3.2) subject to the initial condition $y(t_0) = y_0$, exists throughout any interval about $t = t_0$ in which the functions p and g are continuous.

  On the other hand, for a nonlinear initial value problem satisfying the hypotheses of Theorem 3.5, the interval in which a solution exists may be difficult to determine. This is because it is not so easy to determine the solution $y = \phi(t)$ of a nonlinear equation unlike a linear equation.

- **General Solution.** Another way in which linear and nonlinear equations differ concerns the concept of general solution.

  For a first order linear equation it is possible to obtain a solution containing one arbitrary constant, from which all possible solutions follow by specifying values for this constant as we have seen in the previous lecture.

  For nonlinear equations this is not the case; even though a solution containing an arbitrary constant may be found, there may be other solutions that cannot be obtained by giving values to this constant.
- **Implicit Solutions.** The solution for an initial value problem of a first order linear equation provides an *explicit* formula for the solution $y = \phi(t)$.

  However for a nonlinear equation, the solution is *implicit* in nature, of the form $F(t, y) = 0$.

3.3. Exact Equations and Integrating Factors

In this section we look at a different class of nonlinear equations known as **exact** equations for which there is also a well-defined method of solution. We define an exact equation in the next theorem along with another result.

Theorem 3.10. *Let the functions M, N, M_y, and N_x, where subscripts denote partial derivatives, be continuous in the rectangular region R : $\alpha < x < \beta, \gamma < y < \delta$. Then*

$$M(x, y) + N(x, y)y' = 0 \tag{3.11}$$

is an **exact** *differential equation in R if and only if*

$$M_y(x, y) = N_x(x, y) \tag{3.12}$$

at each point of R. That is, there exists a function ψ satisfying

$$\psi_x(x, y) = M(x, y), \qquad \psi_y(x, y) = N(x, y), \tag{3.13}$$

if and only if M and N satisfy Eq. (3.12).

To find the expression for the solution to the equation (3.11) we see that,

$$M(x, y) + N(x, y)y' = \frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y}\frac{dy}{dx} = \frac{d}{dx}\psi(x, y) \tag{3.14}$$

Therefore Eq. (3.11) becomes

$$\frac{d}{dx}\psi(x, y) = 0. \tag{3.15}$$

Hence the solution to Eq. (3.11) is given implicitly by

$$\psi(x, y) = C. \tag{3.16}$$

for an arbitrary constant C. Let's illustrate the above theorem in the next example.

Example 3. Determine whether the equation is exact. If it is exact, find the solution.

$$(3x^2 - 2xy + 2)\,dx + (6y^2 - x^2 + 3)\,dy = 0. \tag{3.17}$$

Solution 3. $M(x, y) = 3x^2 - 2xy + 2$ and $N(x, y) = 6y^2 - x^2 + 3$. Therefore $M_y(x, y) = -2x$ and $N_x(x, y) = -2x$. Since they are the same, hence Eq. (3.17) is exact. Thus there is a $\psi(x, y)$ such that

$$\psi_x(x, y) = 3x^2 - 2xy + 2.$$

$$\psi_y(x, y) = 6y^2 - x^2 + 3.$$

Integrating the first of these equations with respect to x, we obtain

$$\psi(x, y) = x^3 - x^2y + 2x + h(y). \tag{3.18}$$

Differentiating the above equation with respect to y, we obtain,

$$\psi_y(x, y) = -x^2 + h'(y).$$

Setting $\psi_y = N$ gives

$$-x^2 + h'(y) = 6y^2 - x^2 + 3.$$

Thus $h'(y) = 6y^2 + 3$ and $h(y) = 2y^3 + 3y$. The constant of integration can be omitted since any solution of the preceding equation is satisfactory. Substituting for $h(y)$ in Eq. (3.18) gives

$$\psi(x, y) = x^3 - x^2y + 2x + 6y^2 - x^2 + 3. \tag{3.19}$$

Hence solutions of Eq. (3.17) are given implicitly by

$$x^3 - x^2y + 2x + 6y^2 - x^2 + 3 = C. \tag{3.20}$$

A valid question to ask now is what happens when the initial equation isn't exact. In that situation it is sometimes possible to convert it into an exact equation by multiplying the equation by a suitable integrating factor $\mu(x, y)$.

Unfortunately even though integrating factors are powerful tools for solving differential equations, in practice they can be found only in special cases. The most important situations in which simple integrating factors can be found occur when μ is a function of only one of the variables x or y, instead of both.

If $(M_y - N_x)/N$ is a function of x only, then there is an integrating factor μ that also depends on x and it satisfies the differential equation

$$\frac{d\mu}{dx} = \frac{M_y - N_x}{N}\mu \tag{3.21}$$

If $(N_x - M_y)/M$ is a function of y only, then there is an integrating factor μ that also depends on y and it satisfies the differential equation

$$\frac{d\mu}{dy} = \frac{N_x - M_y}{M}\mu \tag{3.22}$$

Finally we look into an example in which the equation is not exact to begin with but is made exact by multiplying with an integrating factor.

Example 4. Find an integrating factor for the equation

$$(3xy + y^2) + (x^2 + xy)y' = 0. \tag{3.23}$$

and then solve the equation.

Solution 4. Here $M(x, y) = 3xy + y^2$ and $N(x, y) = x^2 + xy$. $M_y \neq N_x$, therefore the differential equation isn't exact. We compute $(M_y - N_x)/N$ and find that

$$\frac{M_y - N_x}{N} = \frac{3x + 2y - (2x + y)}{x^2 + xy} = \frac{1}{x}.$$

Thus there is an integrating factor μ that is a function of x only, and it satisfies the differential equation

$$\frac{d\mu}{dx} = \frac{\mu}{x} \tag{3.24}$$

Solving the above differential equation we have $\mu(x) = x$. Multiplying Eq. (3.23) by this integrating factor, we obtain

$$(3x^2y + xy^2) + (x^3 + x^2y)y' = 0. \tag{3.25}$$

This equation is exact. Therefore there exists a function $\psi(x, y)$, such that

$$\psi_x(x, y) = 3x^2y + xy^2.$$
$$\psi_y(x, y) = x^3 + x^2y.$$

Integrating the first of these equations with respect to x, we obtain

$$\psi(x, y) = x^3y + \frac{x^2y^2}{2} + h(y). \tag{3.26}$$

Differentiating the above equation with respect to y, we have

$$\psi_y(x,y) = x^3 + x^2y + h'(y).$$

Setting $\psi_y = N$ gives

$$x^3 + x^2y + h'(y) = x^3 + x^2y.$$

Therefore $h'(y) = 0$, hence $h(y) = C = 0$. We can choose this constant to be zero since any solution of the preceding equation is satisfactory. Substituting for $h(y)$ in Eq. (3.26) gives

$$\psi(x,y) = x^3y + \frac{x^2y^2}{2}. \tag{3.27}$$

Hence solutions of Eq. (3.23) are given implicitly by

$$x^3y + \frac{x^2y^2}{2} = C. \tag{3.28}$$

First and Second Order Differential Equations Lecture 4

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4.1. Outline of Lecture

- The Existence and the Uniqueness Theorem
- Homogeneous Equations with Constant Coefficients

4.2. The Existence and the Uniqueness Theorem

We have looked at the existence and uniqueness theorem for nonlinear equations in the previous lecture. However verifying the theorem especially for nonlinear equations require solving the initial value problem. In general, finding a solution is not feasible because there is no method of solving the differential equation that applies in all cases.

Therefore for the general case, it is necessary to adopt an indirect approach that demonstrates the existence of a solution. The heart of this method is the construction of a sequence of functions that converges to a limit function satisfying the initial value problem, although the members of the sequence individually do not.

We note that it is sufficient to consider the problem in which the initial point is the origin; that is we consider the problem

$$y' = f(t, y), \qquad y(0) = 0. \tag{4.1}$$

If some other initial point is given, then we can always make a preliminary change of variables, corresponding to a translation of the coordinate axes, that will take the given point to the origin. We can thus modify the existence and uniqueness theorem in the following way.

Theorem 4.2. *If f and $\partial f/\partial y$ are continuous in a rectangle $R : |t| \leq a, |y| \leq b$, then there is some interval $|t| \leq h \leq a$ in which there exists a unique solution $y = \phi(t)$ of the initial value problem (4.1).*

For the method of proof, it is necessary to transform the initial value problem (4.1) into a more convenient form. If we suppose temporarily that there is a differentiable function $y = \phi(t)$ that satisfies the initial value problem, then $f[t, \phi(t)]$ is a continuous function of t only. Hence we can integrate $y' = f(t, y)$ from the initial point $t = 0$ to an arbitrary value of t, obtaining

$$\phi(t) = \int_0^t f[s, \phi(s)]\, ds \tag{4.3}$$

where we have made use of the initial condition $\phi(0) = 0$. The above equation is called an **integral equation**.

One method of showing that the integral equation (4.3) has a unique solution is known as the **method of successive approximations** or Picard's **iteration method**.

We start by choosing an initial function ϕ_0. The simplest choice is

$$\phi_0(t) = 0 \tag{4.4}$$

then ϕ_0 at least satisfies the initial condition in Eq. (4.1), although presumable not the differential equation. The next approximation ϕ_1 is obtained by substituting $\phi_0(s)$ for $\phi(s)$ in the right side of Eq. (4.3) and calling the result of this operation $\phi_1(t)$. Thus

$$\phi_1(t) = \int_0^t f[s, \phi_0(s)]\, ds. \tag{4.5}$$

Similarly, ϕ_2 is obtained from ϕ_1,

$$\phi_2(t) = \int_0^t f[s, \phi_1(s)]\, ds. \tag{4.6}$$

and, in general,

$$\phi_{n+1}(t) = \int_0^t f[s, \phi_n(s)]\, ds. \tag{4.7}$$

In this manner we generate a sequence of functions $\{\phi_n\} = \phi_0, \phi_1, \dots, \phi_n, \dots$. We look into this infinite sequence in the next example.

Example 1. Solve the initial value problem

$$y' = ty + 1, \qquad y(0) = 0 \tag{4.8}$$

by the method of successive approximations.

Solution 1. If $y = \phi(t)$ is the solution then the corresponding integral equation is

$$\phi(t) = \int_0^t (s\phi(s) + 1)\, ds \tag{4.9}$$

If the initial approximation is $\phi_0(t) = 0$, it follows that

$$\phi_1(t) = \int_0^t (s\phi_0(s) + 1)\, ds = \int_0^t ds = t. \tag{4.10}$$

Similarly,

$$\phi_2(t) = \int_0^t (s\phi_1(s) + 1)\, ds = \int_0^t (s^2 + 1)\, ds = \frac{t^3}{3} + t. \tag{4.11}$$

and

$$\phi_3(t) = \int_0^t (s\phi_2(s) + 1)\, ds = \int_0^t (\frac{s^4}{3} + s^2 + 1)\, ds = \frac{t^5}{3 \cdot 5} + \frac{t^3}{3} + t. \tag{4.12}$$

Equations (4.10), (4.11), and (4.12) suggest that

$$\phi_n(t) = t + \frac{t^3}{3} + \frac{t^5}{3 \cdot 5} + \ldots + \frac{t^{2n-1}}{3 \cdot 5 \cdots (2n-1)} \tag{4.13}$$

for each $n \geq 1$, and this result can be established by mathematical induction (Try it!).

It follows from Eq. (4.13) that $\phi_n(t)$ is the nth partial sum of the infinite series

$$\sum_{k=1}^{\infty} \frac{t^{2k-1}}{3 \cdot 5 \cdots (2k-1)} \tag{4.14}$$

hence $\lim_{n\to\infty} \phi_n(t)$ exists if and only if the series (4.14) converges. Applying the ratio test, we see that, for each t,

$$\left| \frac{t^{2k+1} \cdot 3 \cdot 5 \cdots (2k-1)}{3 \cdot 5 \cdots (2k+1) \cdot t^{2k-1}} \right| = \frac{t^2}{2k+1} \to 0 \text{ as } k \to \infty \tag{4.15}$$

Thus the series (4.14) converges for all t, and its sum $\phi(t)$ is the limit of the sequence $\{\phi_n(t)\}$. We can verify by direct substitution that $\phi(t) = \sum_{k=1}^{\infty} \frac{t^{2k-1}}{3 \cdot 5 \cdots (2k-1)}$ is a solution of the integral equation (4.14).

4.3. Homogeneous Equations with Constant Coefficients

We now shift our focus to second order equations. A second order ordinary differential equation has the form

$$\frac{d^2y}{dt^2} = f(t, y, \frac{dy}{dt}), \tag{4.16}$$

where f is some given function. Equation (4.16) is said to be **linear** if the function f has the form

$$f(t, y, \frac{dy}{dt}) = g(t) - p(t)\frac{dy}{dt} - q(t)y, \tag{4.17}$$

where g, p, and q are specified functions of the independent variable t but do not depend on y. In this case we usually rewrite Eq. (4.16) as

$$y'' + p(t)y' + q(t)y = g(t), \tag{4.18}$$

Instead of Eq. (4.18), we often see the equation

$$P(t)y'' + Q(t)y' + R(t)y = G(t). \tag{4.19}$$

If $P(t) \neq 0$, we can divide Eq. (4.19) by $P(t)$ and thereby obtain Eq. (4.18) with

$$p(t) = \frac{Q(t)}{P(t)}, \qquad q(t) = \frac{R(t)}{P(t)}, \qquad g(t) = \frac{G(t)}{P(t)}. \tag{4.20}$$

If Eq. (4.16) is not of the form (4.18) or (4.19), then it is called **nonlinear**.

An initial value problem consists of a differential equation such as Eq. (4.16), or (4.18) together with a pair of initial conditions

$$y(t_0) = y_0, \qquad y'(t_0) = y'_0 \tag{4.21}$$

where y_0 and y'_0 are given numbers prescribing values for y and y' at the initial point t_0. Since we have a second order differential equation therefore, roughly speaking two integrations are required to find a solution and each integration introduces an arbitrary constant. Hence we have two initial conditions.

A second order linear equation is said to be **homogeneous** if the term $g(t)$ in Eq. (4.18) is zero for all t. Otherwise the equation is called **nonhomogeneous**. Therefore a homogeneous equation is of the form

$$y'' + p(t)y' + q(t)y = 0 \tag{4.22}$$

In this section we will concentrate our attention on equations in which the functions P, Q and R are constants. In this case Eq. (4.19) becomes

$$ay'' + by' + cy = 0 \tag{4.23}$$

where a, b and c are given constants.

We now see how we can solve the above equation. We start by seeking exponential solutions of the form $y = e^{rt}$, where r is a parameter to be determined. Then it follows that $y' = re^{rt}$ and $y'' = r^2e^{rt}$. By substituting these expressions for y, y', and y'' in Eq. (4.23) we obtain

$$(ar^2 + br + c)e^{rt} = 0 \tag{4.24}$$

Since $e^{rt} \neq 0$,

$$ar^2 + br + c = 0 \tag{4.25}$$

Equation (4.25) is called the **characteristic equation** for the differential equation (4.23). Solving this quadratic equation gives us two roots r_1 and r_2. In this section we consider the case when both r_1 and r_2 are real and $r_1 \neq r_2$. Then the two solutions are $y_1(t) = e^{r_1 t}$ and $y_2(t) = e^{r_2 t}$. Therefore

$$y = c_1 y_1(t) + c_2 y_2(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t} \tag{4.26}$$

is also a solution of Eq. (4.23) for arbitrary constants c_1 and c_2. We look into an example below which illustrates the method.

Example 2. Find the solution of the initial value problem

$$y'' + 5y' + 6y = 0, \qquad y(0) = 2, \qquad y'(0) = 3. \tag{4.27}$$

Solution 2. We assume that $y = e^{rt}$, and it then follows that r must be a root of the characteristic equation

$$r^2 + 5r + 6 = (r+2)(r+3) = 0. \tag{4.28}$$

Thus the possible values of r are $r_1 = -2$ and $r_2 = -3$; the general solution of Eq. (4.27) is

$$y = c_1 e^{-2t} + c_2 e^{-3t}. \tag{4.29}$$

To satisfy the first initial condition, we set $t = 0$ and $y = 2$ in Eq. (4.29); thus c_1 and c_2 must satisfy

$$c_1 + c_2 = 2. \tag{4.30}$$

To use the second initial condition, we must first differentiate Eq. (4.29). This gives $y' = -2c_1 e^{-2t} - 3c_2 e^{-3t}$. Then, setting $t = 0$ and $y' = 3$, we obtain

$$-2c_1 - 3c_2 = 3. \tag{4.31}$$

By solving Eqs. (4.30) and (4.31), we find that $c_1 = 9$ and $c_2 = -7$. Therefore the solution of the initial value problem (4.27) is

$$y = 9e^{-2t} - 7e^{-3t} \tag{4.32}$$

Note that as $t \to \infty$, the solution $y \to 0$. In general as t increases, the magnitude of the solution either tends to zero (when both exponents are negative) or else grows rapidly (when at least one exponent is positive).

Second Order Differential Equations
Lecture 5

Dibyajyoti Deb

5.1. Outline of Lecture

- Solution of Linear Homogeneous Equations; the Wronskian
- Complex roots of the Characteristic Equation

5.2. Solution of Linear Homogeneous Equations; the Wronskian

In this lecture we provide a clearer picture of the structure of the solutions of all second order linear homogeneous equations using results from previous lectures. We will be asking some basic questions about second order linear homogeneous equations and answer them with the help of some theorems. Before doing that, let's define the notion of a *differential operator*.

Let p and q be continuous functions on an open interval I. Then for any twice differentiable function ϕ on I, we define the **differential operator** L by the equation

$$L[\phi] = \phi'' + p\phi' + q\phi. \tag{5.1}$$

Note that $L[\phi]$ is a function on I. The value of $L[\phi]$ at a point t is

$$L[\phi](t) = \phi''(t) + p(t)\phi'(t) + q(t)\phi(t). \tag{5.2}$$

In this lecture we study the second order linear homogeneous equation $L[\phi](t) = 0$. Since $y = \phi(t)$, we will usually write this equation in the form

$$L[y] = y'' + p(t)y' + q(t)y = 0. \tag{5.3}$$

With Eq. (5.3) we associate a set of initial conditions

$$y(t_0) = y_0, \qquad y'(t_0) = y_0', \tag{5.4}$$

where t_0 is any point in the interval I, and y_0 and y_0' are given real numbers.

The questions that we would like to ask include,

1. Does the initial value problem (5.3), (5.4) always have a solution.
2. If it has a solution then, does it have more than one.
3. Can anything be said about the form and structure of the solutions.

The first two questions are answered with the following theorem.

Theorem 5.5. *(Existence and Uniqueness Theorem)*
Consider the initial value problem

$$y'' + p(t)y' + q(t)y = g(t), \qquad y(t_0) = y_0, \quad y'(t_0) = y_0', \tag{5.6}$$

where p, q, and g are continuous on an open interval I that contains the point t_0. Then there is exactly one solution $y = \phi(t)$ of this problem, and the solution exists throughout the interval I.

The theorem says three things:

- The initial value problem *has* a solution; in other words; a solution *exists*.
- The initial value problem has *only one* solution; that is the solution is *unique*.
- The solution ϕ is defined *throughout the interval I* where the coefficients are continuous and is at least twice differentiable there.

We see an application of the above theorem in the next example.

Example 1. Determine the longest interval in which the given initial value problem is certain to have a unique twice differentiable solution. Do not attempt to find the solution.

$$(t-1)y'' - 3ty' + 4y = \sin t, \qquad y(-2) = 2, \quad y'(-2) = 1 \tag{5.7}$$

Solution 1. If the given differential equation is written in the form of Eq. (5.6), then $p(t) = -3t/(t-1)$, $q(t) = 4/(t-1)$, and $g(t) = \sin t/(t-1)$. The only point of discontinuity of the coefficient is $t = 1$. Therefore, the longest open interval, containing the initial point $t = -2$, in which all the coefficients are continuous is $-\infty < t < 1$. Therefore, this is the longest interval in which the above theorem guarantees that the solution exists.

We look into this next theorem, which provides a way of finding more solutions, starting from two.

Theorem 5.8. *(Principle of Superposition)*
If y_1 and y_2 are two solutions of the differential equation,

$$L[y] = y'' + p(t)y' + q(t)y = 0,$$

then the linear combination $c_1y_1 + c_2y_2$ is also a solution for any values of the constants c_1 and c_2.

Now to answer our third question regarding the form and structure of the solutions of Eq. (5.3), we begin by examining whether the constants c_1 and c_2 from the theorem can be chosen so as to satisfy the initial conditions (5.4). These initial conditions require c_1 and c_2 to satisfy the equations

$$c_1y_1(t_0) + c_2y_2(t_0) = y_0, \tag{5.9}$$

$$c_1y_1'(t_0) + c_2y_2'(t_0) = y_0'. \tag{5.10}$$

The determinant of the coefficients of the above system is

$$W = \begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix} = y_1(t_0)y_2'(t_0) - y_1'(t_0)y_2(t_0). \tag{5.11}$$

If $W \neq 0$, then Eqs. (5.9), (5.10) have a unique solution (c_1, c_2) regardless of the values of y_0 and y_0'. On the other hand, if $W = 0$, then the same equations have no solution unless y_0 and y_0' satisfy a certain additional condition; in this case there are infinitely many solutions.

The determinant W is call the **Wronskian determinant**, or simply the **Wronskian**, of the solutions y_1 and y_2. We use the next theorem for this new result.

Theorem 5.12. *Suppose that y_1 and y_2 are two solutions of Eq. (5.3)*

$$L[y] = y'' + p(t)y' + q(t)y = 0,$$

and that the initial conditions (5.4)

$$y(t_0) = y_0, y'(t_0) = y_0'$$

are assigned. Then it is always possible to choose the constants c_1, c_2 so that

$$y = c_1y_1(t) + c_2y_2(t)$$

satisfies the differential equation (5.3) and the initial conditions (5.4) if and only if the Wronskian

$$W = y_1y_2' - y_1'y_2$$

is not zero at t_0.

The previous theorem gives us a way of constructing infinite number of solutions starting from two solutions y_1 and y_2, whose Wronskian is not zero at the initial point t_0. The next theorem finally answers our third question about the form and structure of the solution of Eq. (5.3).

Theorem 5.13. *Suppose that y_1 and y_2 are two solutions of Eq. (5.3)*

$$L[y] = y'' + p(t)y' + q(t)y = 0,$$

Then the family of solutions

$$y = c_1y_1(t) + c_2y_2(t)$$

with arbitrary coefficients c_1 and c_2 includes every solution of Eq. (5.3) if and only if there is a point t_0 where the Wronskian of y_1 and y_2 is not zero.

Theorem 5.13 states that, if and only if the Wronskian of y_1 and y_2 is not everywhere aero, then the linear combination $c_1y_1+c_2y_2$ contains all solutions of Eq. (5.3). Is is therefore natural to call the expression

$$y = c_1y_1(t) + c_2y_2(t)$$

with arbitrary constant coefficients the **general solution** of Eq. (5.3). The solutions y_1 and y_2 are said to form a **fundamental set of solutions** of Eq. (5.3) if and only if their Wronskian is nonzero.

We look at an application of the above theorem in the next example.

Example 2. Show that $y_1(t) = t^2$ and $y_2(t) = t^{-1}$ are fundamental solutions of the differential equation

$$t^2y'' - 2y = 0 \tag{5.14}$$

for $t > 0$.

Solution 2. We can verify that y_1 and y_2 are indeed solutions to Eq. (5.14) by substitution. To check whether they form a pair of fundamental solutions, we find the Wronskian,

$$W = \begin{vmatrix} t^2 & t^{-1} \\ 2t & -1/t^2 \end{vmatrix} = t^2 \cdot \frac{-1}{t^2} - 2t \cdot t^{-1} = -3 \neq 0. \tag{5.15}$$

Since $W \neq 0$, therefore y_1 and y_2 form a fundamental set of solutions and therefore every other solution is of the form $c_1y_1+c_2y_2$ for arbitrary constants c_1 and c_2.

A new question that arises now, is whether a differential equation of the form (5.3) always has a fundamental set of solutions. The following theorem provides an affirmative answer to this question.

Theorem 5.16. *Consider the differential equation (5.3)*

$$L[y] = y'' + p(t)y' + q(t)y = 0.$$

whose coefficients p and q are continuous on some open interval I. Choose some point t_0 in I. Let y_1 be the solution of Eq. (5.3) that also satisfies the initial conditions

$$y(t_0) = 1, \qquad y'(t_0) = 0,$$

and let y_2 be the solution of Eq. (5.3) that satisfies the initial conditions

$$y(t_0) = 0, \qquad y'(t_0) = 1,$$

Then y_1 and y_2 form a fundamental solutions of Eq. (5.3).

The above theorem assures that a fundamental set of solutions always exists. In fact, a differential equation has infinitely many fundamental solutions.

Now let us examine further the properties of the Wronskian of two solutions of a second order linear homogeneous differential equation. The following theorem, gives a simple explicit formula for the Wronskian of any two solutions of any such equation, even if the solutions themselves are not known.

Theorem 5.17. *(Abel' Theorem)*
If y_1 and y_2 are solutions of the differential equation

$$L[y] = y'' + p(t)y' + q(t)y = 0,$$

where p and q are continuous on some open interval I, then the Wronskian $W(y_1, y_2)(t)$ is given by

$$W(y_1, y_2)(t) = ce^{-\int p(t)\,dt}, \tag{5.18}$$

where c is a certain constant that depends on y_1 and y_2, but not on t. Further, $W(y_1, y_2)(t)$ either is zero for all t in I (if $c = 0$) or else is never zero in I (if $c \neq 0$).

The above theorem says that the Wronskian of any two fundamental sets of solutions of the same differential equation can differ only by a multiplicative constant, and that the Wronskian of any fundamental set of solutions can be determined, up to a multiplicative constant, without solving the differential equation.

We apply the above the theorem in the next example.

Example 3. Find the general form of the Wronskian of the equation

$$2t^2y'' + 3ty' - y = 0, \qquad t > 0 \tag{5.19}$$

Solution 3. We write the differential equation in the standard form with the coefficient of y'' equal to 1. Thus we obtain,

$$y'' + \frac{3}{2t}y' - \frac{1}{2t^2}y = 0, \tag{5.20}$$

so $p(t) = 3/2t$. Hence

$$W(y_1, y_2)(t) = ce^{-\int \frac{3}{2t}\,dt} = ce^{-\frac{3}{2}\ln t} = ct^{-3/2}. \tag{5.21}$$

Equation (5.21) gives the Wronskian of any pair of solutions of the differential equation.

5.3. Complex roots of the Characteristic Equation

In the previous lecture we learned how to solve second order linear homogeneous equation with constant coefficients, whose characteristic equation has different real roots.

In this section we look into the same equation

$$ay'' + by' + cy = 0. \tag{5.22}$$

whose characteristic equation

$$ar^2 + br + c = 0. \tag{5.23}$$

has complex roots. Since the roots are conjugate complex numbers, we denote them by

$$r_1 = \lambda + i\mu, \qquad r_2 = \lambda - i\mu, \tag{5.24}$$

where λ and μ are real. The corresponding expressions for the two solutions are given by (Note the two solutions of equation (5.22) is given by $e^{r_1 t}$ and $e^{r_2 t}$.)

$$y_1(t) = e^{(\lambda+i\mu)t}, \qquad y_2(t) = e^{(\lambda-i\mu)t}. \tag{5.25}$$

y_1 and y_2 can also be written as

$$y_1(t) = e^{\lambda t}e^{i\mu t}, \qquad y_2(t) = e^{\lambda t}e^{-\mu t}, \tag{5.26}$$

We would like to see what it means to raise e to a complex power. The answer is provided by an important relation known as Euler's formula.

Euler's Formula. $e^{i\theta} = \cos\theta + i\sin\theta$.

Using Euler's formula we have $y_1(t) = e^{\lambda t}(\cos\mu t + i\sin\mu t)$, and $y_2(t) = e^{\lambda t}(\cos\mu t - i\sin\mu t)$.

However, rather than using the complex-valued solutions $y_1(t)$ and $y_2(t)$, let us seek instead a fundamental set of solutions of Eq. (5.22) that are real-valued. We know that any linear combination of two solutions is also a solution, so let us form the linear combinations $y_1(t)+y_2(t)$ and $y_1(t)-y_2(t)$. In this way we obtain

$$y_1(t)+y_2(t)=2e^{\lambda t}\cos\mu t,\quad y_1(t)+y_2(t)=2ie^{\lambda t}\sin\mu t. \tag{5.27}$$

Dropping the multiplicative constants 2 and $2i$ for convenience, we are left with

$$u(t)=e^{\lambda t}\cos\mu t,\qquad v(t)=e^{\lambda t}\sin\mu t. \tag{5.28}$$

$u(t)$ and $v(t)$ form a fundamental set of solutions since $W(u,v)=\mu e^{2\lambda t}\neq 0$ (since $\mu\neq 0$). Therefore the general solution of Eq. (5.22) is

$$y=c_1e^{\lambda t}\cos\mu t+c_2e^{\lambda t}\sin\mu t, \tag{5.29}$$

where c_1 and c_2 are arbitrary constants. We look into the next example which uses these results.

Example 4. Solve the given initial value problem.

$$y''+4y'+5y=0,\qquad y(0)=1,\ \ y'(0)=0 \tag{5.30}$$

Solution 4. The characteristic equation is $r^2+4r+5=0$ and its roots are $r=-2\pm i$. Thus the general solution of the differential equation is

$$y=c_1e^{-2t}\cos t+c_2e^{-2t}\sin t. \tag{5.31}$$

To apply the initial condition we set $t=0$ in the the above equation; this gives

$$y(0)=c_1=1. \tag{5.32}$$

For the second initial condition we must differentiate Eq. (5.31) and then set $t=0$.

$$y'=-2c_1e^{-2t}\cos t-c_1e^{-2t}\sin t-2c_2e^{-2t}\sin t+c_2e^{-2t}\cos t. \tag{5.33}$$

$$y'(0)=-2c_1+c_2=0. \tag{5.34}$$

Substituting $c_1=1$, we get $c_2=2$. Using these values of c_1 and c_2 in Eq. (5.31), we obtain

$$y=e^{-2t}\cos t+2e^{-2t}\sin t. \tag{5.35}$$

as the solution of the initial value problem (5.30).

A good question to ask now, is how do the graph of the solution look like. The presence of trigonometric factors in the solution makes the graph into an oscillation. The exponential factor determines the nature of the oscillation as follows.

- If $\lambda>0$, then the oscillations increase with time.
- If $\lambda<0$, then the oscillations decrease with time.

- If $\lambda = 0$, then the oscillations stays constant with time.

Since $\lambda = -2$ in the previous example, therefore the oscillations decay with time.

Second Order Differential Equations
Lecture 6

Dibyajyoti Deb

6.1. Outline of Lecture

- Repeated Roots; Reduction of Order
- Nonhomogeneous Equations; Method of Undetermined Coefficients
- Variation of Parameters

6.2. Repeated Roots; Reduction of Order

In the previous lectures we looked at second order linear homogeneous equations with constant coefficients whose characteristic equation has either different real roots or complex roots. Now we look into the final case, when the characteristic equation has repeated roots.

The characteristic equation of the second order linear homogeneous equation

$$ay'' + by' + cy = 0. \tag{6.1}$$

is

$$ar^2 + br + cr = 0. \tag{6.2}$$

When the above equation has repeated roots then its discriminant $b^2 - 4ac$ is zero. Then the roots are

$$r_1 = r_2 = -b/2a. \tag{6.3}$$

Both these roots yield the same solution. In this case we use the method due to D'Alembert to find a different solution. Recall that since $y_1(t)$ is a solution of Eq. (6.1), so is cy_1 for any constant c. The basic idea is to generalize this observation by replacing c by a function $v(t)$ and then trying to determine $v(t)$ so that the product $v(t)y_1(t)$ is also a solution of Eq. (6.1). We demonstrate this method using the next example.

Example 1. Solve the differential equation

$$y'' + 6y' + 9y = 0. \tag{6.4}$$

Solution 1. The characteristic equation is

$$r^2 + 6r + 9 = (r+3)^2 = 0. \tag{6.5}$$

so $r_1 = r_2 = -3$. Therefore one solution is $y_1(t) = e^{-3t}$. Let $y = v(t)y_1(t)$. We substitute $y = v(t)y_1(t)$ in Eq. (6.4) and use the resulting equation to find $v(t)$. Starting with

$$y = v(t)y_1(t) = v(t)e^{-3t}. \tag{6.6}$$

we have

$$y' = v'(t)e^{-3t} - 3v(t)e^{-3t}. \tag{6.7}$$

and

$$y'' = v''(t)e^{-3t} - 6v'(t)e^{-3t} + 9v(t)e^{-3t}. \tag{6.8}$$

By substituting the expressions in Eqs. (6.6), (6.7), (6.8) in Eq. (6.4) and collecting terms, we obtain

$$[v''(t) - 6v'(t) + 9v(t) + 6v'(t) - 18v(t) + 9v(t)]e^{-3t} = 0. \tag{6.9}$$

which simplifies to

$$v''(t) = 0. \tag{6.10}$$

Therefore

$$v'(t) = c_1 \tag{6.11}$$

and

$$v(t) = c_1 t + c_2, \tag{6.12}$$

where c_1 and c_2 are arbitrary constants. Finally substituting for $v(t)$ in Eq. (6.6), we obtain

$$y = c_1 t e^{-3t} + c_2 e^{-3t}. \tag{6.13}$$

The second term on the right side of Eq. (6.13) corresponds to the original solution $y_1(t) = e^{-3t}$, but the first term arises from a second solution, namely $y_2(t) = te^{-3t}$. We can verify that these solutions form a fundamental set by calculating their Wronskian. The Wronskian turns out to be

$$W(y_1, y_2)(t) = e^{-6t} \neq 0. \tag{6.14}$$

The procedure used in the above example can be generalized to a more general equation whose characteristic equation has repeated roots. In general for an equation

$$ay'' + by' + cy = 0 \tag{6.15}$$

the general solution is

$$y = c_1 e^{-bt/2a} + c_2 t e^{-bt/2a}. \tag{6.16}$$

where c_1 and c_2 are arbitrary constants. The geometrical behavior of solutions is similar in this case to that when the roots are real and different. If the exponents are either positive or negative, then the magnitude of the solution grows or decays accordingly; the linear factor t has little significance. However, if the repeated root is zero, then the differential equation is $y'' = 0$ and the general solution is a linear function of t.

6.2.1. Reduction of Order

The method discussed in the earlier section is more generally applicable. Suppose that we know one solution $y_1(t)$, not everywhere zero, of

$$y'' + p(t)y' + q(t)y = 0. \tag{6.17}$$

We can assume the other solution is $v(t)y_1(t)$ and apply the earlier method to find $v(t)$. We illustrate this in the next example.

Example 2. Given that $y_1(t) = t^{-1}$ is a solution of

$$2t^2y'' + 3ty' - y = 0, \qquad t > 0, \tag{6.18}$$

find a fundamental set of solutions.

Solution 2. We set $y = v(t)t^{-1}$, then

$$y' = v't^{-1} - vt^{-2}, \quad y'' = v''t^{-1} - 2v't^{-2} + 2vt^{-3}. \tag{6.19}$$

Substituting for y, y', and y'' in Eq. (6.18) and collecting terms, we obtain

$$2t^2(v''t^{-1} - 2v't^{-2} + 2vt^{-3}) + 3t(v't^{-1} - vt^{-2}) - vt^{-1} \tag{6.20}$$

$$= 2tv'' - v' = 0. \tag{6.21}$$

Therefore we see that Eq. (6.21) is a separable equation, by noting that $v'' = (v')'$

Separating them out makes both side integrable,

$$\int \frac{(v')'}{v'} = \int \frac{1}{2t}. \tag{6.22}$$

$$\ln |v'(t)| = \ln |ct^{1/2}|. \tag{6.23}$$

Therefore

$$v'(t) = ct^{1/2};$$

then

$$v(t) = \frac{2}{3}ct^{3/2} + k.$$

It follows that

$$y = \frac{2}{3}ct^{1/2} + kt^{-1}. \tag{6.24}$$

where c and k are arbitrary constants. The second term on the right side of Eq. (6.24) is a multiple of $y_1(t)$ and can be dropped, but the first term provides a new solution of $y_2(t) = t^{1/2}$. The Wronskian of y_1 and y_2 is

$$W(y_1, y_2)(t) = \frac{3}{2}t^{-3/2}, \quad t > 0. \tag{6.25}$$

Consequently, y_1 and y_2 form a fundamental set of solutions of Eq. (6.18).

6.3. Nonhomogeneous Equations; Method of Undetermined Coefficients

In this section we learn how to solve a special type of the general nonhomogeneous equation, specifically equations of the form

$$ay'' + by' + cy = g(t), \tag{6.26}$$

where a, b, and c are constants and $g(t)$ is a special function of t.

Before embarking on that, we look at two results that describe the structure of solutions of the general nonhomogeneous equation

$$L[y] = y'' + p(t)y' + q(t)y = g(t), \tag{6.27}$$

where p, q, and g are given continuous functions of the open interval I. Let

$$L[y] = y'' + p(t)y' + q(t)y = 0, \tag{6.28}$$

be the homogeneous equation corresponding to Eq. (6.27).

Theorem 6.29. *If Y_1 and Y_2 are two solutions of the nonhomogeneous equation (6.27), then their difference $Y_1 - Y_2$ is a solution of the corresponding homogeneous equation (6.28). If in addition, y_1 and y_2 are a fundamental set of solutions of Eq. (6.28), then*

$$Y_1(t) - Y_2(t) = c_1 y_1(t) + c_2 y_2(t), \tag{6.30}$$

where c_1 and c_2 are constants.

Proof of the above theorem follows from previous lectures and simple algebra and can be found in the text book.

Theorem 6.31. *The general solution of the nonhomogeneous equation (6.27) can be written in the form*

$$y = \phi(t) = c_1 y_1(t) + c_2 y_2(t) + Y(t), \tag{6.32}$$

where y_1 and y_2 are a fundamental set of solutions of the corresponding homogeneous equation (6.28), c_1 and c_2 are arbitrary constants, and Y is some specific solution of the nonhomogeneous equation (6.27).

The proof of Theorem (6.31) follows quickly from the preceding theorem. We can think of Y_1 as arbitrary solution ϕ and Y_2 as the specific solution Y.

Theorem (6.31) states that to solve the nonhomogeneous equation (6.27), we must do three things:

1. Find the general solution $c_1 y_1(t) + c_2 y_2(t)$ of the corresponding homogeneous equation. This solution is sometimes called the complementary solution and denoted by $y_c(t)$.
2. Find some specific solution $Y(t)$ of the nonhomogeneous equation. This solution is sometimes called the particular solution.
3. Add together the functions found in the two preceding steps.

Since we already know how to find $y_c(t)$, for homogeneous equations with constant coefficients, we would therefore like to find a specific solution of the nonhomogeneous equation (6.26) that we mentioned earlier at the beginning of the section.

We do this in this section for some special functions $g(t)$ in Eq. (6.26) using the Method of Undetermined Coefficients. This method requires us to make an initial assumption about the form of the particular solution $Y(t)$, but with the coefficients left unspecified. We then substitute the assumed expression into the equation and attempt to determine the coefficients so as to satisfy that equation. We summarize the method next.

6.3.1. Method of Undetermined Coefficients.

To find the particular solution let us begin with nonhomogeneous equation with constant coefficients

$$ay'' + by' + cy = g(t), \tag{6.33}$$

where a, b, and c are constants.

1. We make sure that the function $g(t)$ in Eq. (6.26) belongs to one of the classes of functions in the next table, that is, it involves nothing more than exponential functions, sines, cosines, polynomials, or sum or products of such functions.
2. If $g(t) = g_1(t) + \cdots + g_n(t)$, that is, if $g(t)$ is a sum of n terms, then we form n subproblems, each of which contains only one of the terms $g_1(t), \ldots, g_n(t)$. The ith subproblem consists of the equation

$$ay'' + by' + cy = g_i(t), \tag{6.34}$$

   where i runs from 1 to n.
3. Depending on $g_i(t)$, we assume the particular solution $Y_i(t)$ according to the next table.

$g_i(t)$	$Y_i(t)$
$P_n(t) = a_0t^n + a_1t^{n-1} + \cdots + a_n$	$A_0t^n + A_1t^{n-1} + \cdots + A_n$
$P_n(t)e^{\alpha t}$	$(A_0t^n + A_1t^{n-1} + \cdots + A_n)e^{\alpha t}$
$P_n(t)e^{\alpha t}\sin\beta t$ or $P_n(t)e^{\alpha t}\cos\beta t$	$(A_0t^n + A_1t^{n-1} + \cdots + A_n)e^{\alpha t}\cos\beta t + (B_0t^n + B_1t^{n-1} + \cdots + B_n)e^{\alpha t}\sin\beta t$

4. If there is any duplication in the assumed form of $Y_i(t)$ with the solutions of the corresponding homogeneous equation, then multiply $Y_i(t)$ by t, or (if necessary) by t^2, so as to remove the duplication. So for instance if we want to find a particular solution of

$$y'' + 4y' + 4y = 6te^{-2t}, \tag{6.35}$$

   our choice of $Y(t)$ would have to be At^2e^{-2t} since te^{-2t} (which we find from the above table) is a solution of the corresponding homogeneous equation of Eq. (6.35).
5. Find a particular solution $Y_i(t)$ for each subproblems. Then the sum $Y_1(t) + \cdots + Y_n(t)$ is a particular solution of the full nonhomogeneous equation (6.26).

Let us look at an example which uses the above method.

Example 3. Find the particular solution of

$$y'' - 3y' - 4y = 2e^{-t} \tag{6.36}$$

Solution 3. The table says that our assumption for $Y(t)$ should be Ae^{-t} for some constant A that is to be determined. However e^{-t} is a solution of the corresponding homogeneous equation of (6.36)

$$y'' - 3y' - 4y = 0 \tag{6.37}$$

Therefore we modify our assumption of $Y(t)$, by multiplying it with t and assume that the particular solution is of the form $Y(t) = Ate^{-t}$. Then

$$Y'(t) = Ae^{-t} - Ate^{-t}, \quad Y''(t) = -2Ae^{-t} + Ate^{-t}.$$

Substituting these expressions for y, y' and y'' in Eq. (6.36), we obtain

$$(-2A - 3A)e^{-t} + (A + 3A - 4A)te^{-t} = 2e^{-t}. \tag{6.38}$$

Hence $-5A = 2$, so $A = -2/5$. Thus a particular solution of Eq. (6.36) is

$$Y(t) = -\frac{2}{5}te^{-t}. \tag{6.39}$$

6.4. Variation of Parameters

In this section we describe another method of finding a particular solution of a non-homogenoeus equation. This method is known as **variation of parameters**. The main advantage of variation of parameters is that it is a general method. Without further adieu, let's look into the general theorem illustrating the method.

Theorem 6.40. *If the functions p, q and g are continuous on an open interval I, and if the functions y_1 and y_2 are a fundamental set of solutions of the homogeneous equation (6.28) corresponding to the non-homogeneous equation (6.27)*

$$y'' + p(t)y' + q(t)y = g(t), \tag{6.41}$$

then a particular solution of Eq. (6.27) is

$$Y(t) = -y_1(t)\int_{t_0}^{t} \frac{y_2(s)g(s)}{W(y_1, y_2)(s)}\,ds + y_2(t)\int_{t_0}^{t} \frac{y_1(s)g(s)}{W(y_1, y_2)(s)}\,ds, \tag{6.42}$$

where t_0 is any conveniently chosen point in I. The general solution is

$$y = c_1y_1(t) + c_2y_2(t) + Y(t). \tag{6.43}$$

As was mentioned earlier, this method is a general method; in principle at least, it can be applied to any equation, and it requires no detailed assumptions about the form of the solution. On the other hand, the method of variation of parameters requires us to evaluate certain integrals involving the nonhomogeneous term in the differential equation, and this may present difficulties.

We dive into an example which uses the above method.

Example 4. The given functions y_1 and y_2 satisfy the corresponding homogeneous equation. Find a particular solution of the given nonhomogeneous equation.

$$t^2y'' - 2y = 3t^2 - 1, \quad t > 0, \quad y_1(t) = t^2, \;\; y_2(t) = t^{-1} \tag{6.44}$$

Solution 4. Writing the above equation in the standard form we have,

$$y'' - \frac{2}{t^2}y = 3 - \frac{1}{t^2} \tag{6.45}$$

Therefore $p(t) = 0, q(t) = -\frac{2}{t^2}$ and $g(t) = 3 - \frac{1}{t^2}$. The thress functions are continuous whenever $t \neq 0$. Therefore we choose $t_0 = 1$. We also have $W(y_1, y_2) = -3$. By the above theorem

$$\begin{aligned} Y(t) &= -t^2 \int_1^t \frac{\frac{1}{s} \cdot (3 - \frac{1}{s^2})}{-3}\, ds + \frac{1}{t} \int_1^t \frac{s^2 \cdot (3 - \frac{1}{s^2})}{-3}\, ds \\ &= \frac{t^2}{3} \int_1^t (\frac{3}{s} - \frac{1}{s^3})\, ds - \frac{1}{3t} \int_1^t (3s^2 - 1)\, ds \end{aligned}$$

Integrating the above expression and using the limits we have

$$Y(t) = \frac{t^2}{3}(3\ln t + \frac{1}{2t^2} - \frac{1}{2}) - \frac{1}{3t}(t^3 - t)$$

After simplification we have

$$Y(t) = t^2 \ln t + \frac{1}{2} - \frac{t^2}{2}$$

Since t^2 is already a solution of the corresponding homogeneous equation, we can ignore it at this moment. Hence the particular solution of Eq. (6.44) is given by

$$Y(t) = t^2 \ln t + \frac{1}{2} \tag{6.46}$$

Higher Order Linear Equations
Lecture 7

Dibyajyoti Deb

7.1. Outline of Lecture

- General Theory of nth Order Linear Equations.
- Homogeneous Equations with Constant Coefficients.

7.2. General Theory of nth Order Linear Equations

An nth order linear differential equation is an equation of the form

$$L[y] = y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_{n-1}(t)y' + p_n(t)y = g(t). \tag{7.1}$$

Since the equation involves the $n$the derivative of y, therefore to obtain a unique solution, it is necessary to specify n initial conditions

$$y(t_0) = y_0, \quad y'(t_0) = y'_0, \quad \ldots, y^{(n-1)}(t_0) = y_0^{(n-1)}. \tag{7.2}$$

The mathematical theory associated with Eq. (7.1) is completely analogous to that for the second order linear equation. Therefore we simply state the results for the nth order problem.

Theorem 7.3. *If the functions $p_1, p_2, \ldots, p_n$, and g are continuous on the open interval I, then there exists exactly one solution $y = \phi(t)$ of the differential equation (7.1) that also satisfies the initial conditions (7.2). The solution exists throughout the interval I.*

7.2.1. The Homogeneous Equation.

As in the corresponding second order problem, we first discuss the homogeneous equation

$$L[y] = y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_{n-1}(t)y' + p_n(t)y = 0. \tag{7.4}$$

If the functions $y_1, y_2, \dots, y_n$ are solutions of Eq. (7.4), then it follows by direct computation that the linear combination

$$y = c_1 y_1(t) + c_2 y_2(t) + \cdots + c_n y_n(t), \tag{7.5}$$

where $c_1, \dots, c_n$ are arbitrary constants, is also a solution of Eq. (7.4).

We define the Wronskian of the solutions $y_1, \dots, y_n$ by the determinant

$$W(y_1, \dots, y_n) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix} \tag{7.6}$$

Theorem 7.7. *If the functions $p_1, p_2, \dots, p_n$ are continuous on the open interval I, if the functions $y_1, y_2, \dots, y_n$ are solutions of Eq. (7.4), and if $W(y_1, y_2, \dots, y_n)(t) \neq 0$ for at least one point in I, then every solution of Eq. (7.4) can be expresses as a linear combination of the solutions $y_1, y_2, \dots, y_n$.*

A set of solutions $y_1, \dots, y_n$ of Eq. (7.4) whose Wronskian is nonzero is referred to as a **fundamental set of solutions**. Since all solutions of Eq. (7.4) are of the form (7.5), we use the term **general solution** to refer to any arbitrary linear combination of any fundamental set of solutions of Eq. (7.4).

7.2.2. Linear Dependence and Independence.

We now explore the relationship between fundamental sets of solutions and the concept of linear independence.

The functions $f_1, f_2, \dots, f_n$ are said to be **linearly dependent** on an interval I if there exists a set of constants $k_1, k_2, \dots, k_n$, not all zero, such that

$$k_1 f_1(t) + k_2 f_2(t) + \cdots + k_n f_n(t) = 0 \tag{7.8}$$

for all t in I. The functions $f_1, \dots, f_n$ are said to be **linearly independent** on I if they are not linearly dependent there. We look into an example.

Example 1. Determine whether the functions $f_1(t) = 1, f_2(t) = 2 + t, f_3(t) = 3 - t^2$, and $f_4(t) = 4t + t^2$ are linearly independent or dependent on any interval I.

Solution 1. We form the linear combination

$$\begin{aligned} k_1 f_1(t) + k_2 f_2(t) + k_3 f_3(t) + k_4 f_4(t) &= k_1 + k_2(2+t) + k_3(3-t^2) + k_4(4t+t^2) \\ &= (k_1 + 2k_2 + 3k_3) + (k_2 + 4k_4)t + (-k_3 + k_4)t^2. \end{aligned}$$

This expression is zero throughout an interval provided that

$$k_1 + 2k_2 + 3k_3 = 0, \quad k_2 + 4k_4 = 0, \quad -k_3 + k_4 = 0.$$

These three equations, with four unknowns, have many nontrivial solutions. For instance, if $k_4 = 1$, then $k_3 = 1, k_2 = -4$, and $k_1 = 5$. Thus the given functions are linearly dependent on every interval.

We now present the theorem describing the relation between linear independence and fundamental sets of solutions.

Theorem 7.9. *If $y_1(t), \ldots, y_n(t)$ is a fundamental set of solutions of Eq. (7.4)*

$$L[y] = y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_{n-1}(t)y' + p_n(t)y = 0 \tag{7.10}$$

on an interval I, then $y_1(t), \ldots, y_n(t)$ are linearly independent on I. Conversely, if $y_1(t), \ldots, y_n(t)$ are linearly independent solutions of Eq. (7.4) on I, then they form a fundamental set of solutions of I.

7.2.3. The Nonhomogeneous Equation.

Consider the nonhomogeneous equation (7.1)

$$L[y] = y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_{n-1}(t)y' + p_n(t)y = g(t). \tag{7.11}$$

It follows that any solution of the above equation can be written as

$$y = c_1y_1(t) + c_2y_2(t) + \cdots + c_ny_n(t) + Y(t), \tag{7.12}$$

where $y_1, \ldots, y_n$ is fundamental set of solutions of the corresponding homogeneous equation and Y is some particular solution of the nonhomogeneous equation (7.1). The linear combination (7.12) is called the general solution of the nonhomogeneous equation (7.1).

7.3. Homogeneous Equations with Constant Coefficients

Consider the nth order linear homogeneous differential equation

$$L[y] = a_0y^{(n)} + a_1y^{(n-1)} + \cdots + a_{n-1}y' + a_ny = 0, \tag{7.13}$$

where $a_0, a_1, \ldots, a_n$ are real constants. From our knowledge of second order linear equations with constant coefficients, it is natural to anticipate that $y = e^{rt}$ is a solution of Eq. (7.13) for suitable values of r. Indeed,

$$L[e^{rt}] = e^{rt}(a_0r^n + a_1r^{n-1} + \cdots + a_{n-1}r + a_n) = e^{rt}Z(r) \tag{7.14}$$

for all r, where

$$Z(r) = a_0r^n + a_1r^{n-1} + \cdots + a_{n-1}r + a_n. \tag{7.15}$$

The polynomial $Z(r)$ is called the **characteristic polynomial**, and the equation $Z(r) = 0$ is the **characteristic equation** of the differential equation (7.13). A polynomial of degree n has n zeros, say $r_1, r_2, \ldots, r_n$, some of which may be equal; hence we can write the characteristic polynomial in the form

$$Z(r) = a_0(r - r_1)(r - r_2)\cdots(r - r_n). \tag{7.16}$$

Now we look at all the three possibilities of the nature of the roots.

7.3.1. Real and Unequal Roots.

If the roots of the characteristic equation are real and no two are equal, then we have n distinct solutions $e^{r_1 t}, e^{r_2 t}, \ldots, e^{r_n t}$ of Eq. (7.13). If these functions are linearly independent(check Wronskian), then the general solution of Eq. (7.13) is

$$y = c_1 e^{r_1 t} + c_2 e^{r_2 t} + \cdots + c_n e^{r_n t}. \tag{7.17}$$

7.3.2. Complex Roots.

If the characteristic equation has complex roots, they must occur in conjugate pairs, $\lambda \pm i\mu$, since the coefficients $a_0, \ldots, a_n$ are real numbers. Provided that none of the roots are repeated, the general solution of Eq. (7.13) is still of the form (7.17). Similar to the second order equation, we can replace the complex valued solutions $e^{(\lambda+i\mu)t}$ and $e^{(\lambda-i\mu)t}$ by the real-valued solutions

$$e^{\lambda t}\cos \mu t, \quad e^{\lambda t}\sin \mu t \tag{7.18}$$

7.3.3. Repeated Roots.

If the roots of the characteristic equation are not distinct, that is if some of the roots are repeated, then we have to look at the multiplicity of the root. For an equation of order n, if a root of $Z(r) = 0$, say $r = r_1$, has multiplicity s (where $s \leq n$), then

$$e^{r_1 t}, \quad te^{r_1 t}, \quad t^2 e^{r_1 t}, \quad \ldots, \quad t^{s-1} e^{r_1 t} \tag{7.19}$$

are corresponding solutions of Eq. (7.13).

If a complex root $\lambda + i\mu$ is repeated s times, the complex conjugates $\lambda - i\mu$ is also repeated s times. Corresponding to these $2s$ complex valued solutions, we can find $2s$ real valued solutions by noting that the real and imaginary parts of $e^{(\lambda+i\mu)t}, te^{(\lambda+i\mu)t}, \ldots, t^{s-1}e^{(\lambda+i\mu)t}$ are also linearly independent solutions:

$$e^{\lambda t}\cos \mu t, \quad e^{\lambda t}\sin \mu t, \quad te^{\lambda t}\cos \mu t, \quad te^{\lambda t}\sin \mu t,$$
$$\ldots, \quad t^{s-1}e^{\lambda t}\cos \mu t, \quad t^{s-1}e^{\lambda t}\sin \mu t.$$

Let's look into an example below.

Example 2. Find the general solution of the given differential equation.

(7.20) $$y''' - 3y'' + 7y' - 5y = 0.$$

Solution 2. The characteristic equation of the above differential equation is given by

(7.21) $$Z(r) = r^3 - 3r^2 + 7r - 5 = 0$$

Substituting $r = 1$, it can be verified that $Z(1) = 0$, hence $r = 1$ is a root of $Z(r)$. Since $(r-1)$ is a factor of $Z(r)$, hence by the Factor Theorem, the other factor can be found by dividing $Z(r)$ by $(r-1)$. The other factor is $r^2 - 2r + 5$ whose roots are $1 \pm 2i$. Hence the three roots of Eq. (7.20) are

(7.22) $$e^t, \quad e^t \cos 2t, \quad e^t \sin 2t.$$

Therefore the general solution of Eq. (7.20) is given by

(7.23) $$y = c_1 e^t + c_2 e^t \cos 2t + c_3 e^t \sin 2t.$$

for arbitrary constants c_1, c_2, c_3.

Higher Order Linear Equations
Lecture 8

Dibyajyoti Deb

8.1. Outline of Lecture

- The Method of Undetermined Coefficients.
- The Method of Variation of Parameters.

8.2. The Method of Undetermined Coefficients.

A particular solution Y of the nonhomogeneous nth order linear equation with constant coefficients

$$L[y] = a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = g(t) \tag{8.1}$$

can be obtained by the method of undetermined coefficients, provided that $g(t)$ is of an appropriate form. This mimics the method of undetermined coefficients for second order nonhomogeneous equations (See Lecture 6). Thus if $g(t)$ is a polynomial $A_0 t^m + A_1 t^{m-1} + \cdots + A_m$, an exponential function $e^{\alpha t}$, a sine function $\sin \beta t$, cosine function $\cos \beta t$, or a combination of them, then our assumed solution is also a suitable combination of polynomials, exponentials, and so forth, multiplied by a number of undetermined constants. The constants are then determined by substituting the assumed expression into Eq. (8.1).

The main difference in using this method for higher order equations stems from the fact that roots of the characteristic polynomial equation may have multiplicity greater than 2. Consequently, terms proposed for the nonhomgeneous part of the solution may need to be multiplied by higher powers of t to make them different from terms in the solution of the corresponding homogeneous equation. We look at this case in the example below.

Example 1. Find the general solution of

$$y''' - 3y'' + 3y' - y = 4e^t. \tag{8.2}$$

Solution 1. The characteristic polynomial for the homogeneous equation corresponding is

$$r^3 - 3r^2 + 3r - 1 = (r-1)^3, \tag{8.3}$$

Since the root 1 is repeated three times, therefore the general solution of the homogeneous equation is

$$y_c(t) = c_1 e^t + c_2 t e^t + c_3 t^2 e^t. \tag{8.4}$$

To find a particular solution $Y(t)$, we start by assuming that $Y(t) = Ae^t$. However since e^t, te^t, and t^2e^t are all solutions of the homogeneous equation, we must multiply this initial choice by t^3. Thus our final assumption is that $Y(t) = At^3e^t$, where A is an undetermined coefficient.

We differentiate $Y(t)$ three times, substitute for y and its derivative in Eq. (8.2), and collect terms in the resulting equation. In this way we obtain

$$6Ae^t = 4e^t. \tag{8.5}$$

Therefore $A = \frac{2}{3}$ and the particular solution is

$$Y(t) = \frac{2}{3} t^3 e^t. \tag{8.6}$$

Therefore the general solution of Eq. (8.2) is given by

$$y = c_1 e^t + c_2 t e^t + c_3 t^2 e^t + \frac{2}{3} t^3 e^t. \tag{8.7}$$

8.3. The Method of Variation of Parameters.

The method of variation of parameters for determining a particular solution of the nonhomogeneous nth order linear differential equation

$$L[y] = y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_{n-1}(t)y' + p_n(t)y = g(t) \tag{8.8}$$

is a direct extension of the method for the second order differential equation that was covered in Lecture 6.

We look into the general theorem which illustrates the method.

Theorem 8.9. *If the functions $p_1, \ldots, p_n$ and g are continuous on an open interval I, and if the functions $y_1, y_2, \ldots, y_n$ are a fundamental set of solutions of the homogeneous equation corresponding to the nonhomogeneous equation (8.8), then a particular solution of Eq. (8.8) is*

$$Y(t) = \sum_{m=1}^{n} y_m(t) \int_{t_0}^{t} \frac{g(s)W_m(s)}{W(s)}\, ds, \tag{8.10}$$

where $W(t) = W(y_1, y_2, \ldots, y_n)(t)$, t_0 is any conveniently chosen point in I and W_m is the determinant obtained from W by replacing the mth column by the column $(0, 0, \ldots, 0, 1)$. The general solution is

$$y = c_1 y_1(t) + c_2 y_2(t) + \ldots + c_n y_n(t) + Y(t). \tag{8.11}$$

Let's look into an example which uses the above theorem.

Example 2. Use the method of variation of parameters to determine the general solution of the given differential equation.

$$y''' - y' = t \tag{8.12}$$

Solution 2. The characteristic polynomial of the corresponding homogeneous equation of Eq. (8.12) is

$$r^3 - r = 0 \tag{8.13}$$

The roots of this equation are $0, 1$ and -1. Therefore the solutions of the homogeneous equation are $1, e^t$ and e^{-t}.

$$W(1, e^t, e^{-t}) = \begin{vmatrix} 1 & e^t & e^{-t} \\ 0 & e^t & -e^{-t} \\ 0 & e^t & e^{-t} \end{vmatrix} = 2 \tag{8.14}$$

Hence $y_1(t) = 1, y_2(t) = e^t$ and $y_3(t) = e^{-t}$ form a fundamental set of solution. Therefore

$$W_1(t) = \begin{vmatrix} 0 & e^t & e^{-t} \\ 0 & e^t & -e^{-t} \\ 1 & e^t & e^{-t} \end{vmatrix} = -2 \tag{8.15}$$

$$W_2(t) = \begin{vmatrix} 1 & 0 & e^{-t} \\ 0 & 0 & -e^{-t} \\ 0 & 1 & e^{-t} \end{vmatrix} = e^{-t} \tag{8.16}$$

$$W_3(t) = \begin{vmatrix} 1 & e^t & 0 \\ 0 & e^t & 0 \\ 0 & e^t & 1 \end{vmatrix} = e^t \tag{8.17}$$

We choose $t_0 = 0$. Using the above theorem

$$Y(t) = 1 \cdot \int_0^t \frac{s(-2)}{2}\, ds + e^t \int_0^t \frac{se^{-s}}{2}\, ds + e^{-t} \int_0^t \frac{se^s}{2}\, ds \tag{8.18}$$

We evaluate the last two integrals using integration by parts. This gives

$$Y(t) = -\frac{t^2}{2} - 1 + \frac{e^{-t}}{2}. \tag{8.19}$$

Since 1 and e^{-t} are already solutions of the homogeneous equation, therefore the particular solution of Eq. (8.12) is $Y(t) = -\frac{t^2}{2}$.

Therefore the general solution of Eq. (8.12) is

$$y = c_1 + c_2e^t + c_3e^{-t} - \frac{t^2}{2}. \tag{8.20}$$

The Laplace Transform
Lecture 9

Dibyajyoti Deb

9.1. Outline of Lecture

- Definition of the Laplace Transform.
- Solution of Initial Value Problems.

9.2. Definition of the Laplace Transform.

9.2.1. Improper Integrals.

We look into a brief overview of improper integrals. An improper integral over an unbounded interval is defined as a limit of integrals over finite intervals; thus

$$\int_a^\infty f(t)\,dt = \lim_{A\to\infty}\int_a^A f(t)\,dt, \tag{9.1}$$

where A is a positive real number. If the integral from a to A exists for each $A > a$, and if the limit as $A \to \infty$ exists, then the improper integral is said to **converge** to that limiting value. Otherwise the integral is said to **diverge**, or fail to exist.

We look at couple of examples.

Example 1. Let $f(t) = e^{ct}, t \geq 0$, where c is a real nonzero constant. Then

$$\int_0^\infty e^{ct}\,dt = \lim_{A\to\infty}\int_0^A e^{ct}\,dt = \lim_{A\to\infty}\left.\frac{e^{ct}}{c}\right|_0^A \tag{9.2}$$

$$= \lim_{A\to\infty}\frac{1}{c}(e^{cA}-1). \tag{9.3}$$

It follows that the improper integral converges to the value $-1/c$ if $c < 0$ and diverges if $c > 0$. If $c = 0$, then $f(t)$ is the constant function with value 1, and the integral again diverges.

Example 2. Let $f(t) = 1/t, t \geq 1$. Then

$$\int_1^\infty \frac{dt}{t} = \lim_{A\to\infty} \int_1^A \frac{dt}{t} = \lim_{A\to\infty} \ln A. \tag{9.4}$$

Since $\lim_{A\to\infty} \ln A = \infty$, the improper integral diverges.

9.2.2. The Laplace Transform.

Among the tools that are very useful for solving linear differential equations are **integral transforms**. An integral transform is a relation of the form

$$F(s) = \int_\alpha^\beta K(s,t) f(t)\, dt, \tag{9.5}$$

where $K(s,t)$ is a given function, called the **kernel** of the transformation, and the limits of integration α and β are also given. The relation (9.5) transforms the function f into another function F, which is called the transform of f.

The Laplace transform of f, which we will denote by $\mathcal{L}\{f(t)\}$ or by $F(s)$, is defined by the equation

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\, dt, \tag{9.6}$$

whenever the integral converges. The general idea in using the Laplace transform to solve a differential equation is as follows:

1. Use the relation (9.6) to transform an initial value problem for an unknown function f in the t-domain into a simpler problem (indeed, an algebraic problem) for F in the s-domain.
2. Solve this algebraic problem to find F.
3. Recover the desired function f from its transform F.

We look into an example where we find the Laplace transform.

Example 3. Find the Laplace transform of the function $f(t) = t$.

Solution 1.

$$\mathcal{L}\{t\} = \int_0^\infty e^{-st} t\, dt = \lim_{A\to\infty} \int_0^A e^{-st} t\, dt \tag{9.7}$$

We use integration by parts,

$$(9.8)\quad = \lim_{A\to\infty}\left[\frac{te^{-st}}{-s}\Bigg|_0^A + \frac{1}{s}\int_0^A e^{-st}\,dt\right] = \lim_{A\to\infty}\frac{Ae^{-sA}}{-s} - \frac{1}{s^2}\lim_{A\to\infty} e^{-st}\Bigg|_0^A$$

$$(9.9)\quad = -\frac{1}{s^2}\lim_{A\to\infty}\left[e^{-sA}-1\right] = \frac{1}{s^2}, \quad s>0.$$

Therefore

$$(9.10)\quad \mathcal{L}\{t\} = \frac{1}{s^2}, \quad s>0.$$

An important result before we wrap up this section. The Laplace transform is a **linear operator**, that is, if f_1 and f_2 are two functions whose Laplace transforms exist for $s > a_1$ and $s > a_2$, respectively. Then, for s greater than the maximum of a_1 and a_2,

$$(9.11)\quad \mathcal{L}\{c_1 f_1(t) + c_2 f_2(t)\} = c_1\mathcal{L}\{f_1(t)\} + c_2\mathcal{L}\{f_2(t)\}.$$

9.3. Solution of Initial Value Problems.

In this section we show how the Laplace transform can be used to solve initial value problems for linear differential equations with constant coefficients. We present a theorem which we will use extensively in this section

Theorem 9.12. *Suppose that the functions $f, f', \ldots, f^{(n-1)}$ are continuous and that $f^{(n)}$ is piecewise continuous on any interval $0 \le t \le A$. Suppose further that there exist constants K, a and M such that $|f(t)| \le Ke^{at}, |f'(t)| \le Ke^{at}, \ldots, |f^{(n-1)}(t)| \le Ke^{at}$ for $t \ge M$. Then $\mathcal{L}\{f^{(n)}(t)\}$ exists for $s > a$ and is given by*

$$(9.13)\quad \mathcal{L}\{f^{(n)}(t)\} = s^n\mathcal{L}\{f(t)\} - s^{n-1}f(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0).$$

We look into an example where we solve an initial value problem using the method of Laplace transform.

Example 4. Find the solution of the differential equatio

$$(9.14)\quad y'' + y = \cos 2t.$$

satisfying the initial conditions

$$(9.15)\quad y(0) = 1, \quad y'(0) = 0.$$

Solution 2. Taking the Laplace transform of the differential equation by using Theorem (9.12), we have

$$(9.16)\quad s^2\mathcal{L}\{y\} - sy(0) - y'(0) + \mathcal{L}\{y\} = \mathcal{L}\{\cos 2t\}.$$

Using the table from the text book we have $\mathcal{L}\{\cos 2t\} = s/(s^2+4)$. Therefore

$$s^2\mathcal{L}\{y\} - s + \mathcal{L}\{y\} = \frac{s}{s^2+4} \tag{9.17}$$

$$\mathcal{L}\{y\}(s^2+1) = \frac{s}{s^2+4} + s$$

$$\mathcal{L}\{y\} = \frac{s}{(s^2+4)(s^2+1)} + \frac{s}{(s^2+1)} \tag{9.18}$$

Just like Laplace transform, the inverse Laplace transform is also a linear operator. Therefore

$$y = \mathcal{L}^{-1}\{\frac{s}{(s^2+4)(s^2+1)}\} + \mathcal{L}^{-1}\{\frac{s}{(s^2+1)}\} \tag{9.19}$$

Using partial fractions, we can write $s/(s^2+4)(s^2+1)$ in the form

$$\frac{s}{(s^2+4)(s^2+1)} = \frac{As+B}{s^2+4} + \frac{Cs+D}{s^2+1} \tag{9.20}$$

Multiplying both sides of the above equation by $(s^2+4)(s^2+1)$ we have

$$s = (As+B)(s^2+1) + (Cs+D)(s^2+4). \tag{9.21}$$

Expanding the right side we have

$$s = s^3(A+C) + s^2(B+D) + s(A+4C) + (B+4D). \tag{9.22}$$

Equation coefficients of like powers of s, we have

$$A+C=0, \quad B+D=0, \quad A+4C=1, \quad B+4D=0. \tag{9.23}$$

Consequently, $A = -\frac{1}{3}, B = 0, C = \frac{1}{3}, D = 0$, from which it follows that

$$y = \mathcal{L}^{-1}\{-\frac{1}{3}(\frac{s}{s^2+4}) + \frac{1}{3}(\frac{s}{s^2+1})\} + \mathcal{L}\{\frac{s}{s^2+1}\}. \tag{9.24}$$

$$= \mathcal{L}^{-1}\{-\frac{1}{3}(\frac{s}{s^2+4})\} + \mathcal{L}^{-1}\{\frac{1}{3}(\frac{s}{s^2+1})\} + \mathcal{L}\{\frac{s}{s^2+1}\}. \tag{9.25}$$

Using the table from the text book we have

$$y = -\frac{1}{3}\cos 2t + \frac{1}{3}\cos t + \cos t = -\frac{1}{3}\cos 2t + \frac{4}{3}\cos t. \tag{9.26}$$

Therefore the solution of the given initial value problem is

$$y = -\frac{1}{3}\cos 2t + \frac{4}{3}\cos t. \tag{9.27}$$

The Laplace Transform
Lecture 10

Dibyajyoti Deb

10.1. Outline of Lecture

- Step Functions.
- Differential Equations with Discontinuous Forcing Functions.

10.2. Step Functions.

In this section we look at functions which have jump discontinuities. Differential equations whose right side is a function of this type frequently arise in the analysis of the flow of current in electric circuits or the vibrations of mechanical systems. We develop some additional properties of Laplace transform in this section and the next which will help us in the solution of such problems.

To deal with functions with jump discontinuities we introduce a function known as the **unit step function** or **Heaviside function**. This function is denoted by u_c and is defined for $c \geq 0$ by

$$u_c(t) = \begin{cases} 0, & t < c, \\ 1, & t \geq c \end{cases}$$

We want to write a piecewise continuous function in a more "compact" manner with the help of the step function in order to find its Laplace transform. We look into an example below where we do this.

Example 1. Consider the function

$$f(t) = \begin{cases} 2, & 0 \leq t < 2, \\ 5, & 2 \leq t < 5, \\ -3, & 5 \leq t < 9, \\ 3, & t \geq 9 \end{cases}$$

Express $f(t)$ in terms of $u_c(t)$.

Solution 1. We start with the function $f_1(t) = 2$ which agrees with $f(t)$ on $[0, 2)$. To produce the jump of three units (going from 2 to 5) at $t = 2$, we add $3u_2(t)$ to $f_1(t)$, obtaining

$$f_2(t) = 2 + 3u_2(t). \tag{10.1}$$

which agrees with $f(t)$ on $[0, 7)$. The negative jump of eight units (going from 5 to -3) at $t = 5$ corresponds to adding $-8u_5(t)$, which gives

$$f_3(t) = 2 + 3u_2(t) - 8u_5(t). \tag{10.2}$$

Finally to get the positive jump of six units (going from -3 to 3) at $t = 9$, we add $6u_9(t)$. Thus we obtain

$$f(t) = 2 + 3u_2(t) - 8u_5(t) + 6u_9(t). \tag{10.3}$$

The Laplace transform of u_c is easily determined

$$\mathcal{L}\{u_c(t)\} = \int_0^\infty e^{-st}u_c(t)\,dt = \int_0^c e^{-st}u_c(t)\,dt + \int_c^\infty e^{-st}u_c(t)\,dt \tag{10.4}$$

$$= \int_c^\infty e^{-st}\,dt = \frac{e^{-cs}}{s}, \quad s > 0. \tag{10.5}$$

For a given function f defined for $t \geq 0$, we will often want to consider the related function g defined by

$$y = g(t) = \begin{cases} 0, & t < c, \\ f(t-c), & t \geq c, \end{cases} \tag{10.6}$$

In terms of the unit step function we can write $g(t)$ in the convenient form

$$g(t) = u_c(t)f(t-c). \tag{10.7}$$

We look into the first theorem where we find the Laplace transform of $g(t)$.

Theorem 10.8. *If $F(s) = \mathcal{L}\{f(t)\}$ exists for $s > a \geq 0$, and if c is a positive constant, then*

$$\mathcal{L}\{u_c(t)f(t-c)\} = e^{-cs}\mathcal{L}\{f(t)\} = e^{-cs}F(s), \quad s > a. \tag{10.9}$$

Conversely, if $f(t) = \mathcal{L}^{-1}\{F(s)\}$, then

$$u_c(t)f(t-c) = \mathcal{L}^{-1}\{e^{-cs}F(s)\}. \tag{10.10}$$

Proof. Check the text book. □

We look into an example which uses this theorem.

Example 2. Find the inverse transform of

$$G(s) = \frac{2e^{-2s}}{s^2 - 4} \tag{10.11}$$

Solution 2.

$$\mathcal{L}^{-1}\{G(s)\} = \mathcal{L}^{-1}\{\frac{2e^{-2s}}{s^2 - 4}\} = \mathcal{L}^{-1}\{e^{-2s}\frac{2}{s^2 - 4}\} = \mathcal{L}^{-1}\{e^{-2s}F(s)\} \tag{10.12}$$

where $F(s) = \dfrac{2}{s^2 - 4}$. By the converse of Theorem 10.8,

$$\mathcal{L}^{-1}\{e^{-2s}\frac{2}{s^2 - 4}\} = u_2(t)f(t-2) \tag{10.13}$$

where $f(t) = \mathcal{L}^{-1}\{\dfrac{2}{s^2 - 4}\} = \sinh 2t$.

Therefore $f(t-2) = \sinh(2(t-2)) = \sinh(2t-4)$. Hence

$$\mathcal{L}^{-1}\{G(s)\} = u_2(t)\sinh(2t-4). \tag{10.14}$$

We look into another theorem that contains another very useful property of Laplace transform that is somewhat analogous to the previous theorem.

Theorem 10.15. *If $F(s) = \mathcal{L}\{f(t)\}$ exists for $s > a \geq 0$, and if c is a constant, then*

$$\mathcal{L}\{e^{ct}f(t)\} = F(s-c), \quad s > a + c. \tag{10.16}$$

Conversely, if $f(t) = \mathcal{L}^{-1}\{F(s)\}$, then

$$e^{ct}f(t) = \mathcal{L}^{-1}\{F(s-c)\}. \tag{10.17}$$

Proof.

$$\mathcal{L}\{e^{ct}f(t)\} = \int_0^\infty e^{-st} \cdot e^{ct}f(t)\,dt = \int_0^\infty e^{-(s-c)t}f(t)\,dt = F(s-c).$$

□

We look into an example which uses the above theorem.

Example 3. Find the inverse transform of

$$G(s) = \frac{3!}{(s-2)^4} \tag{10.18}$$

Solution 3.

$$G(s) = \frac{3!}{(s-2)^4} = F(s-2) \tag{10.19}$$

where $F(s) = \frac{3!}{s^4}$.

$\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\{\frac{3!}{s^4}\} = t^3 = f(t)$ by the converse of Theorem 10.15.

By Theorem 10.15,

$$\mathcal{L}^{-1}\{G(s)\} = \mathcal{L}^{-1}\{F(s-2)\} = e^{2t}t^3. \tag{10.20}$$

10.3. Differential Equations with Discontinuous Forcing Functions.

In this section we turn our attention to solving differential equations in which the nonhomogeneous term is discontinuous. We look into an example below.

Example 4. Find the solution of the given initial value problem.

$$y'' + y = f(t); \quad y(0) = 0, \quad y'(0) = 1, \quad f(t) = \begin{cases} 1, & 0 \leq t \leq 3\pi \\ 0, & 3\pi \leq t < \infty \end{cases} \tag{10.21}$$

Solution 4. Using the step function

$$f(t) = 1 - u_{3\pi}(t). \tag{10.22}$$

Therefore the equation becomes

$$y'' + y = 1 - u_{3\pi}(t). \tag{10.23}$$

$$\mathcal{L}\{y''\} + \mathcal{L}\{y\} = \mathcal{L}\{1\} - \mathcal{L}\{u_{3\pi}(t)\} \tag{10.24}$$

$$s^2\mathcal{L}\{y\} - sy(0) - y'(0) + \mathcal{L}\{y\} = \frac{1}{s} - \frac{e^{-3\pi s}}{s}$$

$$s^2\mathcal{L}\{y\} - 1 + \mathcal{L}\{y\} = \frac{1}{s} - \frac{e^{-3\pi s}}{s}$$

$$\mathcal{L}\{y\}(s^2+1) = \frac{1}{s} - \frac{e^{-3\pi s}}{s} + 1$$

$$\mathcal{L}\{y\} = \frac{1}{s(s^2+1)} - \frac{e^{-3\pi s}}{s(s^2+1)} + \frac{1}{s^2+1}$$

$$y = \mathcal{L}^{-1}\{\frac{1}{s(s^2+1)}\} - \mathcal{L}^{-1}\{\frac{e^{-3\pi s}}{s(s^2+1)}\} + \mathcal{L}^{-1}\{\frac{1}{s^2+1}\}$$

We use partial fractions(Check Lecture Notes 9) to write

$$\frac{1}{s(s^2+1)} = \frac{1}{s} - \frac{s}{s^2+1} \tag{10.25}$$

Therfore

$y = \mathcal{L}^{-1}\{\frac{1}{s}\} - \mathcal{L}^{-1}\{\frac{s}{s^2+1}\} - \mathcal{L}^{-1}\{e^{-3\pi s}\frac{1}{s}\} - \mathcal{L}^{-1}\{e^{-3\pi s}\frac{s}{s^2+1}\} + \mathcal{L}^{-1}\{\frac{1}{s^2+1}\}$

By Theorem 10.8,

$$\mathcal{L}^{-1}\{e^{-3\pi s}\frac{1}{s}\} = u_{3\pi}(t)f(t-3\pi). \tag{10.26}$$

and

$$\mathcal{L}^{-1}\{e^{-3\pi s}\frac{s}{s^2+1}\} = u_{3\pi}(t)g(t-3\pi). \tag{10.27}$$

where $f(t) = \mathcal{L}^{-1}\{\frac{1}{s}\} = 1$ and $g(t) = \mathcal{L}^{-1}\{\frac{s}{s^2+1}\} = \cos t$. Therefore

$$\mathcal{L}^{-1}\{e^{-3\pi s}\frac{1}{s}\} = u_{3\pi}(t)f(t-3\pi) = u_{3\pi}(t).$$

and

$$\mathcal{L}^{-1}\{e^{-3\pi s}\frac{s}{s^2+1}\} = u_{3\pi}(t)g(t-3\pi) = u_{3\pi}(t)\cos(t-3\pi).$$

Therefore the solution to the differential equation is

$$y = 1 - \cos t - u_{3\pi}(t) - u_{3\pi}(t)\cos(t-3\pi) + \sin t. \tag{10.28}$$

Since $\cos(t-3\pi) = \cos t$, therefore

$$y = 1 - \cos t - u_{3\pi}(t) - u_{3\pi}(t)\cos t + \sin t. \tag{10.29}$$

The Laplace Transform
Lecture 11

Dibyajyoti Deb

11.1. Outline of Lecture

- Impulse Functions.
- The Convolution Integral.

11.2. Impulse Functions.

In this section we look at functions of impulsive nature, i.e. forces of large magnitude that act over short time intervals. A mechanical interpretation might be the use of a hammer to strike an object or the striking of a baseball with a bat. We would like to have a mathematical way of representing these types of forces.

To do this, we will introduce a new "function", the Dirac delta "function".

11.2.1. The Dirac delta

We define the Dirac delta such that it satisfies the following properties.

Definition 11.1. The Dirac delta at $t = t_0$, denoted by $\delta(t - t_0)$, satisfies the following properties:

(1) $\delta(t - t_0) = 0, \quad t \neq t_0,$

(2) $\displaystyle\int_{t_0-\tau}^{t_0+\tau} \delta(t - t_0)\, dt = 1, \quad$ for any $\tau > 0$,

(3) $\displaystyle\int_{t_0-\tau}^{t_0+\tau} f(t)\delta(t - t_0)\, dt = f(c), \quad$ for any $\tau > 0$.

We can think of $\delta(t - t_0)$ as having an "infinite" value at $t = t_0$, so that its total energy is 1, all concentrated at that point. So the Dirac delta can be thought of as an instantaneous impulse at $t = t_0$.

11.2.2. The Laplace transform of the Dirac delta

To solve initial value problems involving the Dirac delta, we need to know its Laplace transform. By the third property of the Dirac delta,

$$\mathcal{L}\{\delta(t-t_0)\} = \int_0^\infty e^{-st}\delta(t-t_0)\,dt = e^{-t_0 s}, \quad c > 0. \tag{11.2}$$

also,

$$\mathcal{L}\{f(t)\delta(t-t_0)\} = \int_0^\infty e^{-st}f(t)\delta(t-t_0)\,dt = f(t_0)e^{-t_0 s}, \quad c > 0. \tag{11.3}$$

We look into an example below,

Example 1. Find the solution of the given initial value problem.

$$y'' - y = -20\delta(t-3), \qquad y(0) = 1, y'(0) = 0. \tag{11.4}$$

Solution 1.

$$\mathcal{L}\{y''\} - \mathcal{L}\{y\} = -20\mathcal{L}\{\delta(t-3)\}$$
$$s^2\mathcal{L}\{y\} - sy(0) - y'(0) - \mathcal{L}\{y\} = -20e^{-3s}$$
$$s^2\mathcal{L}\{y\} - s - \mathcal{L}\{y\} = -20e^{-3s}$$
$$\mathcal{L}\{y\}(s^2-1) = -20e^{-3s} + s.$$
$$\mathcal{L}\{y\} = \frac{-20e^{-3s}}{s^2-1} + \frac{s}{s^2-1}$$
$$y = -20\mathcal{L}^{-1}\{e^{-3s}\frac{1}{s^2-1}\} + \mathcal{L}^{-1}\{\frac{s}{s^2-1}\}$$

By # 8 and # 13 from the table on Page 317, we have

$$y = -20u_3(t)f(t-3) + \cosh t.$$

where $f(t) = \mathcal{L}^{-1}\{\frac{1}{s^2-1}\} = \sinh t$.

Therefore the solution to the initial value problem is

$$y = -20u_3(t)\sinh(t-3) + \cosh t. \tag{11.5}$$

11.3. The Convolution Integral.

If a Laplace transform $H(s)$ can be written as the product of two other transforms $F(s)$ and $G(s)$, then a good question to ask is whether the same is true for their inverse Laplace transform, i.e. whether $\mathcal{L}^{-1}\{H(s)\} = \mathcal{L}^{-1}\{F(s)\}\mathcal{L}^{-1}\{G(s)\}$. However, this is not the case. In this section we look into exact relation between the inverse Laplace transforms.

Theorem 11.6. *If* $F(s) = \mathcal{L}\{f(t)\}$ *and* $G(s) = \mathcal{L}\{g(t)\}$ *both exist for* $s > a \geq 0$, *then*

$$H(s) = F(s)G(s) = \mathcal{L}\{h(t)\}, \quad s > a, \tag{11.7}$$

where

$$h(t) = \int_0^t f(t-\tau)g(\tau)\,d\tau = \int_0^t f(\tau)g(t-\tau)\,d\tau. \tag{11.8}$$

The function h *is known as the* **convolution** *of* f *and* g*; the integral in Eq. (11.8) are known as convolution integrals.*

It is convenient to emphasize that the convolution integral can be thought of as a "generalized product" by writing

$$h(t) = (f \star g)(t). \tag{11.9}$$

The convolution $f \star g$ has many of the properties of ordinary multiplication. For example, it is relatively simple to show that

$$f \star g = g \star f \tag{11.10}$$

$$f \star (g_1 + g_2) = f \star g_1 + f \star g_2 \tag{11.11}$$

$$(f \star g) \star h = f \star (g \star h) \tag{11.12}$$

$$f \star 0 = 0 \star f = 0. \tag{11.13}$$

However there are other properties of ordinary multiplication that the convolution integral does not have such as

$$f \star 1 \neq f. \tag{11.14}$$

We look into an example below.

Example 2. Find the inverse Laplace transform of

$$H(s) = \frac{a}{s^2(s^2+s^2)}. \tag{11.15}$$

Solution 2. We can think of

$$H(s) = F(s) \cdot G(s) = \frac{1}{s^2} \cdot \frac{a}{s^2+a^2} \tag{11.16}$$

By # 3 and # 5 of the table on Page 317,

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = t, \text{ and } g(t) = \mathcal{L}^{-1}\{G(s)\} = \sin at. \tag{11.17}$$

By Theorem (11.6), the inverse transform of $H(s)$ is

$$h(t) = \int_0^t (t-\tau)\sin a\tau\,d\tau. \tag{11.18}$$

Using integration by parts, we have

$$\int_0^t (t-\tau)\sin a\tau \, d\tau = \frac{at - \sin at}{a^2}. \tag{11.19}$$

Therefore

$$\mathcal{L}^{-1}\{H(s)\} = \frac{at - \sin at}{a^2}. \tag{11.20}$$

Series Solutions of Second Order Linear Equations Lecture 12

Dibyajyoti Deb

12.1. Outline of Lecture

- Review of Power Series.
- Series Solutions near an Ordinary Point, Part I.

12.2. Review of Power Series.

Our goal from the very beginning has been to find the solution of a general second order equation without any restrictions to the coefficients or the forcing functions. In this regard we have given a systematic procedure for constructing solutions if the equation has constant coefficients. To deal with the much larger class of equations that have variable coefficients, it is necessary to extend our search for solutions beyond the familiar elementary functions of calculus. The principal tool that we need is the representation of a given function by a power series.

In this section we start by looking at some basic properties of power series.

12.2.1. Quick review of Power Series.

1. A power series $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ is said to converge at a point x if

$$\lim_{m\to\infty}\sum_{n=0}^{m} a_n(x-x_0)^n \tag{12.1}$$

exists for that x. The series certainly converges for $x = x_0$; it may converge for all x, or it may converge for some values of x and nor for others.

2. The series $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ is said to converge absolutely at a point x if the series

$$|\sum_{n=0}^{\infty} a_n(x-x_0)^n| = \sum_{n=0}^{\infty} |a_n||(x-x_0)^n| \tag{12.2}$$

converges. A thing to note is that absolute convergences implies convergence but not the other way around.

3. One of the most useful tests for the absolute convergence of a power series is the **ratio test**. If $a_n \neq 0$, and if, for a fixed value of x,

$$\lim_{n\to\infty} \left| \frac{a_{n+1}(x-x_0)^{n+1}}{a_n(x-x_0)^n} \right| = |x-x_0| \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = |x-x_0|L, \tag{12.3}$$

then the power series converges absolutely at that value of x if $|x-x_0|L < 1$ and diverges if $|x-x_0|L > 1$. If $|x-x_0|L = 1$, the test is inconclusive.

4. If the power series $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ converges for $x = x_1$, it converges absolutely for $|x-x_0| < |x_1-x_0|$; and if it diverges at $x = x_1$, it diverges for $|x-x_0| > |x_1-x_0|$.

5. There is a nonnegative number ρ, called the **radius of convergence**, such that $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ converges absolutely for $|x-x_0| < \rho$ and diverges for $|x-x_0| > \rho$. For a series that converges only at x_0, we define ρ to be zero; for a series that converges for all x, we say that ρ is infinite. If $\rho > 0$, then the interval $|x-x_0| < \rho$ is called the **interval of convergence**.

6. The value of a_n is given by

$$a_n = \frac{f^{(n)}(x_0)}{n!}. \tag{12.4}$$

The series is called the Taylor series for the function f about $x = x_0$.

7. A function f that has a Taylor series expansion about $x = x_0$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n, \tag{12.5}$$

with radius of convergence $\rho > 0$, is said to be **analytic** at $x = x_0$.

We look into couple of examples below.

Example 1. For which values of x does the power series

$$\sum_{n=1}^{\infty}(-1)^{n+1}n(x-3)^n \tag{12.6}$$

converge?

Solution 1. We use the ratio test. We have

$$\lim_{n\to\infty}\left|\frac{(-1)^{n+2}(n+1)(x-3)^{n+1}}{(-1)^{n+1}n(x-3)^n}\right| = |x-3|\lim_{n\to\infty}\frac{n+1}{n} = |x-3|. \tag{12.7}$$

According to statement 3, the series converges absolutely for $|x-3|<1$, or $2<x<4$, and diverges for $|x-3|>1$, or $x>4$ and $x<2$. To find what happens at $x=2$ and $x=4$ we substitute these values back into the original power series to see that both the series diverges since the nth term does not approach zero as $n\to\infty$. Hence the series converges in the open interval $(2,4)$.

Example 2. Determine the radius of convergence of the power series

$$\sum_{n=0}^{\infty}\frac{n}{2^n}x^n \tag{12.8}$$

Solution 2. We apply the ratio test. We have

$$\lim_{n\to\infty}\left|\frac{(n+1)x^{(n+1)}}{2^{n+1}}\frac{2^n}{nx^n}\right| = \frac{|x|}{2}\lim_{n\to\infty}\frac{n+1}{n} = \frac{|x|}{2}. \tag{12.9}$$

Thus the series converges absolutely for $|x|<2$, or $-2<x<2$, and diverges for $|x|>2$ or $x>2$ and $x<-2$. At the endpoints $x=2$ and $x=-2$, the series diverges since the nth term does not approach zero as $n\to\infty$. The radius of convergence of the power series is $\rho=2$.

12.3. Series Solutions near an Ordinary Point, Part I.

In previous sections we described methods of solving second order linear differential equations with constant coefficients. We now consider methods of solving second order linear equations when the coefficients are functions of the independent variable. It is sufficient to consider the homogeneous equation

$$P(x)y'' + Q(x)y' + R(x)y = 0 \tag{12.10}$$

since the procedure for the corresponding nonhomogeneous equation is similar.

For the present, suppose that P, Q and R are polynomials and that they have no common factors. Suppose that we wish to solve Eq. (12.10) in the neighborhood of a point x_0.

A point x_0 such that $P(x_0) \neq 0$ is called an **ordinary point**. Since P is continuous, it follows that there is an interval about x_0 in which $P(x)$ is never zero. In this section we will find series solutions to Eq. (12.10) near an ordinary point x_0.

On the other hand, if $P(x_0) = 0$, then x_0 is called a **singular point** of Eq. (12.10). We look into an example directly.

Example 3. Find a series solution of the equation

$$y'' - xy' - y = 0 \tag{12.11}$$

Solution 3. Here $P(x) = 1, Q(x) = -x$ and $R(x) = -1$. Hence we could pick out ordinary point to be $x_0 = 0$ and find a solution near this point. We look for a solution in the form of a power series about $x_0 = 0$

$$y = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots = \sum_{n=0}^{\infty} a_n x^n \tag{12.12}$$

and assume that the series converges in some interval $|x| < \rho$. Differentiating Eq. (12.12) term by term yields

$$y' = a_1 + 2a_2 x + \cdots + n a_n x^{n-1} + \cdots = \sum_{n=1}^{\infty} n a_n x^{n-1} \tag{12.13}$$

(12.14)

$$y'' = 2a_2 + 2 \cdot 3 a_3 x + \cdots + n(n-1) a_n x^{n-2} + \cdots = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substituting the series (12.13) and (12.14) for y and y'' and y in (12.11) gives

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - x \sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n = 0 \tag{12.15}$$

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^n$$

Shifting the index of the term on the left and shifting the first term on the right so that both n starts from zero, we have,

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n = \sum_{n=0}^{\infty} na_n x^n + \sum_{n=0}^{\infty} a_n x^n \tag{12.16}$$

Note the goal here is to make the index and the degree of x on all the three summations to be the same. We do this here by making the index n start from zero and the degree of x being n throughout.

Equating the coefficient of x^n from both sides we have,

$$(n+2)(n+1)a_{n+2} = na_n + a_n = (n+1)a_n. \tag{12.17}$$

Simplifying we have our **recurrence relation**,

$$(n+2)a_{n+2} = a_n. \tag{12.18}$$

or

$$a_{n+2} = \frac{a_n}{n+2} \tag{12.19}$$

Since a_{n+2} is given in terms of a_n, the a's are determined in steps of two. Thus a_0 determines a_2, which in turn determines $a_4, \ldots$; a_1 determines a_3 which in turn determines $a_5, \ldots$. For the even numbered coefficients we have

$$a_2 = \frac{a_0}{2}, \quad a_4 = \frac{a_2}{4} = \frac{a_0}{2\cdot 4}, \quad a_6 = \frac{a_4}{6} = \frac{a_0}{2\cdot 4\cdot 6}, \ldots \tag{12.20}$$

These results suggest that in general, if $n = 2k$, then

$$a_n = a_{2k} = \frac{a_0}{2\cdot 4\cdot 6\cdot \ldots \cdot 2k} = \frac{a_0}{2^k k!}, \quad k = 1, 2, 3, \ldots \tag{12.21}$$

Similarly, for the odd-numbered coefficients we have

$$a_3 = \frac{a_1}{3}, \quad a_5 = \frac{a_3}{5} = \frac{a_1}{3\cdot 5}, \quad a_7 = \frac{a_5}{7} = \frac{a_1}{3\cdot 5\cdot 7}, \ldots \tag{12.22}$$

Similarly these results suggest that in general, if $n = 2k+1$, then

$$a_n = a_{2k+1} = \frac{a_1}{3\cdot 5\cdot 7\cdot \ldots \cdot 2k+1} = \frac{2^k k! a_1}{(2k+1)!} \tag{12.23}$$

Substituting these coefficients into Eq. (12.12), we have

$$\begin{aligned} y &= a_0 + a_1 x + \frac{a_0}{2}x^2 + \frac{a_1}{3}x^3 + \frac{a_0}{2^2 2!}x^4 + \frac{2^2 2! a_1}{5!}x^5 + \cdots + \frac{a_0}{2^n n!}x^{2n} + \frac{2^n n! a_1}{(2n+1)!}x^{2n+1} + \cdots \\ &= a_0\left[1 + \frac{1}{2}x^2 + \frac{1}{2^2 2!}x^4 + \cdots + \frac{1}{2^n n!}x^{2n} + \cdots\right] \\ &\quad + a_1\left[x + \frac{1}{3}x^3 + \frac{2^2 2!}{5!}x^5 + \cdots + \frac{2^n n!}{(2n+1)!}x^{2n+1}\right] \end{aligned}$$

$$= a_0 \sum_{n=0}^{\infty} \frac{1}{2^n n!} x^{2n} + a_1 \sum_{n=0}^{\infty} \frac{2^n n!}{(2n+1)!} x^{2n+1}$$

It is easy to see by ratio test that both of these series converge for all x (Check it!).

Series Solutions of Second Order Linear Equations Lecture 13

Dibyajyoti Deb

13.1. Outline of Lecture

- Series Solutions near an Ordinary Point, Part II.
- Euler Equations.

13.2. Series Solutions near an Ordinary Point, Part II.

In the previous lecture, we considered the problem of finding solutions of

$$P(x)y'' + Q(x)y' + R(x)y = 0, \tag{13.1}$$

where P, Q, and R are polynomials, in the neighborhood of an ordinary point x_0. Assuming that Eq. (13.1) does have a solution $y = \phi(x)$ and that ϕ has a Taylor series

$$y = \phi(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n, \tag{13.2}$$

which converges for $|x - x_0| < \rho$, where $\rho > 0$, we found a_n can be determined by directly substituting the series (13.2) for y in Eq. (13.1).

We now consider how we might justify the statement that if x_0 is an ordinary point of Eq. (13.1) then there exists solutions of the form (13.2).

Suppose there is a solution of Eq. (13.1) of the form (13.2). By differentiating Eq. (13.2) m times and setting x equal to x_0 we have

$$m! a_m = \phi^{(m)}(x_0). \tag{13.3}$$

Hence, to compute a_n from the above expression, we need to determine $\phi^{(n)}(x_0)$ for $n = 0, 1, 2, \ldots$.

To compute $\phi^{(n)}(x_0)$, we use the original differential equation (13.1). Since ϕ is a solution of Eq. (13.1), we have

$$P(x)\phi''(x) + Q(x)\phi'(x) + R(x)\phi(x) = 0. \tag{13.4}$$

We can find $\phi''(x)$ from the above equation

$$\phi''(x) = -p(x)\phi'(x) - q(x)\phi(x), \tag{13.5}$$

where $p(x) = Q(x)/P(x)$ and $q(x) = R(x)/P(x)$. Setting x equal to x_0 in Eq. (13.5) gives

$$\phi''(x_0) = -p(x_0)\phi'(x_0) - q(x_0)\phi(x_0). \tag{13.6}$$

From here we can find a_2 since

$$2!a_2 = \phi''(x_0) = -p(x_0)\phi'(x_0) - q(x_0)\phi(x_0). \tag{13.7}$$

It can be easily checked that $\phi'(x_0) = a_1$ and $\phi(x_0) = a_0$. Therefore

$$2!a_2 = \phi''(x_0) = -p(x_0)a_1 - q(x_0)a_0. \tag{13.8}$$

To determine a_3, we differentiate Eq. (13.5) and set x equal to x_0, obtaining

$$3!a_3 = \phi'''(x_0) = -2!p(x_0)a_2 - [p'(x_0) + q(x_0)]a_1 - q'(x_0)a_0. \tag{13.9}$$

As we see from above to compute the remaining a_n's we have to compute infinitely many derivatives of p and q. Unfortunately, this condition is too weak to ensure that we can prove the convergence of the resulting series expansion for $y = \phi(x)$. What is needed is to assume that the functions p and q are *analytic* at x_0.

With this we can generalize the definitions of an ordinary point and singular point of Eq. (13.1) as follows: if the functions $p = Q/P$ and $q = R/P$ are analytic at x_0, then the point x_0 is said to be an **ordinary point** of the differential equation (13.1); otherwise it is a **singular point**.

Now we shift our focus to finding the interval of convergence of the series solution. We look into a theorem which answers the question for a wide class of problems.

Theorem 13.10. *If x_0 is an ordinary point of the differential equation (13.1)*

$$P(x)y'' + Q(x)y' + R(x)y = 0, \tag{13.11}$$

that is, if $p = Q/P$ and $q = R/P$ are analytic at x_0, then the general solution of Eq. (13.1) is

$$y = \sum_{n=0}^{\infty} a_n(x - x_0)^n = a_0y_1(x) + a_1y_2(x), \tag{13.12}$$

where a_0 and a_1 are arbitrary, and y_1 and y_2 are two power series solutions that are analytic at x_0. The solutions y_1 and y_2 form a fundamental set of solutions. Further, the radius of convergence for each of the series solutions y_1 and y_2 is at least as large as the minimum of the radii of convergence of the series for p and q.

We will not prove this theorem, here however, there is an easier way to compute the lower bound of the radius of convergence of the series solution when P, Q and R are polynomials. We present it in the next two results.

13.2.1. Result 1

The ratio of two polynomials, say, Q/P, has a convergent power series expansion about a point $x = x_0$ if $P(x_0) \neq 0$.

13.2.2. Result 2

If any factors common to Q and P have been canceled, then the radius of convergence of the power series of Q/P about the point x_0 is precisely the distance from x_0 to the nearest root of P.

We use these results in the form of an example below.

Example 1. Determine a lower bound for the radius of convergence of the series solution about the given point x_0, for the given differential equation.

$$(x^2 - 2x - 3)y'' + xy' + 4y = 0; \quad x_0 = 4. \tag{13.13}$$

Solution 1. The roots of $P(x) = x^2 - 2x - 3$ are 3 and -1. The nearest root to $x_0 = 4$ is the root 3 and the distance is 1. Hence the lower bound for the radius of convergence of the series solution of the differential equation is 1, i.e. the series solution $\sum_{n=0}^{\infty} a_n(x-4)^n$ converges for at least $|x - 4| < 1$.

13.3. Euler Equations.

In this section we will begin to consider how to solve equations of the form

$$P(x)y'' + Q(x)y' + R(x)y = 0, \tag{13.14}$$

in the neighborhood of a singular point x_0, i.e. where $P(x_0) = 0$. Instead of looking at any general equation, we will only consider a special type of second order equation called the Euler equation in this lecture.

13.3.1. Euler Equations.

A simple differential equation that has a singular point is the **Euler equation**

$$x^2y'' + \alpha xy' + \beta y = 0, \tag{13.15}$$

where α and β are real constants.

To solve a equation of this type our initial assumption of a solution would be $y = x^r$ for any constant r. Substituting back into Eq. (13.15) we have

$$x^2(x^r)'' + \alpha x(x^r)' + \beta x^r = 0.$$
$$x^r[r(r-1) + \alpha r + \beta] = 0.$$

We call the quadratic equation in r

$$r(r-1) + \alpha r + \beta = r^2 + (\alpha - 1)r + \beta = 0 \tag{13.16}$$

the characteristic equation. Based on the roots r_1 and r_2 of Eq. (13.16), we have the following solutions of Eq. (13.15).

- If r_1 and r_2 are real and $r_1 \neq r_2$, the general solution is

$$y = c_1|x|^{r_1} + c_2|x|^{r_2}. \tag{13.17}$$

- If r_1 and r_2 are real and $r_1 = r_2$, the general solution is

$$y = c_1|x|^{r_1} + c_2|x|^{r_1}\ln|x|. \tag{13.18}$$

- If r_1 and r_2 are complex then let $r_1 = \lambda + i\mu$ and $r_2 = \lambda - i\mu$, the general solution is

$$y = c_1|x|^{\lambda}\cos(\mu\ln|x|) + c_2|x|^{\lambda}\sin(\mu\ln|x|). \tag{13.19}$$

  for arbitrary constants c_1 and c_2 which can be determined with initial conditions.

We present an example of an Euler equation below.

Example 2. Determine the general solution of the given differential equation that is valid in any interval not including the singular point.

$$x^2y'' - xy' + y = 0 \tag{13.20}$$

Solution 2. For this Euler equation $\alpha = -1$ and $\beta = 1$. Hence the characteristic equation is

$$r^2 - 2r + 1 = 0 \tag{13.21}$$

whose root 1 is repeated. Hence the general solution is

$$y = c_1|x| + c_2|x|\ln|x|. \tag{13.22}$$

for arbitrary constants c_1 and c_2.