Handout 1
MATH 252

Dibyajyoti Deb

1.1. Antiderivatives

1.1.1. Performance Criteria

(a) Compute the anti-derivative of a basic form (linear combinations of x^n for any rational n, $\sin x$, $\cos x$ and e^x) without use of formulas or a calculator.
(b) Solve an initial value problem.

1.1.2. Indefinite Integral

You have learned about derivatives in previous chapters. A good question to ask at this point would be the "inverse" problem related to it.

Question. Given the derivative, can we find the original function?

Example 1.1. The derivative of a certain function is $-\cos x$. Find one such function.

Solution. So, let's look at it this way. Say $f(x) = -\cos x = F'(x)$. Therefore, we are looking for $F(x)$. So what could be $F(x)$? Forget about the "$-$" sign in front of $\cos x$ for a moment here. Let's think of a function whose derivative is $\cos x$.

$$\frac{d}{dx}(\sin x) = \cos x.$$

The answer is $\sin x$. Now what about the "$-$" that was in front of $\cos x$?

$$\begin{aligned} -\frac{d}{dx}(\sin x) &= -\cos x. \\ \frac{d}{dx}(-\sin x) &= -\cos x. \end{aligned}$$

Thus, $-\sin x$ is one of many functions whose derivative is $-\cos x$. Can you think of other functions whose derivative is also $-\cos x$?

$\square$

The function $F(x)$ that we found out in the previous example has a special name. We call it the **antiderivative** of $f(x)$. Let's define it formally.

Definition 1.2. A function $F(x)$ is an antiderivative of $f(x)$ on (a,b) if $F'(x) = f(x)$ for all $x \in (a,b)$.

Therefore, from Example 1.1 we can say that $-\sin x$ is an antiderivative of $-\cos x$.

Question. Is $-\sin x + 1$ an antiderivate of $-\cos x$?

Answer. Yes, since

$$\frac{d}{dx}(-\sin x + 1) = -\cos x.$$

In fact, it's easy to see that any function $F(x)$ of the form $-\sin x + C$, where C is a constant is an antiderivative of $-\cos x$, since

$$\frac{d}{dx}(-\sin x + C) = -\cos x.$$

$\square$

In general, if $F(x)$ is an antiderivate of $f(x)$, then every other antiderivative of $f(x)$ is of the form $F(x) + C$ for some constant C. The process of finding an antiderivative is called **integration**. Instead of using sentences like "$F(x)$ is an antiderivative of $f(x)$", we can use a simple notation for it.

Definition 1.3. The notation

$$\int f(x)\,dx = F(x) + C,$$

means that $F(x)$ is the antiderivative of $f(x)$ i.e. $F'(x) = f(x)$. We say that $F(x) + C$ is the indefinite integral of $f(x)$. The constant C is called the **constant of integration**. You always need to include this constant C whenever you are finding the indefinite integral of a function.

Thus from Example 1.1,

$$\int -\cos x\,dx = -\sin x + C.$$

1.1.3. Some common indefinite integrals

We look at some common indefinite integrals that we will be using later,

- **Power Rule:** $\displaystyle\int x^n\,dx = \frac{x^{n+1}}{n+1} + C, \qquad \text{for } n \neq -1.$
- $\displaystyle\int \frac{1}{x}\,dx = \ln|x| + C.$
- $\displaystyle\int \sin(ax+b)\,dx = -\frac{1}{a}\cos(ax+b) + C$, for constants $a \neq 0, b$.
- $\displaystyle\int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b) + C$, for constants $a \neq 0, b$.
- $\displaystyle\int e^{kx}\,dx = \frac{1}{k}e^{kx} + C, \qquad \text{for a constant } k \neq 0.$
- **Sum Rule:** $\displaystyle\int (f(x)+g(x))\,dx = \int f(x)\,dx + \int g(x)\,dx$
- **Multiples Rule:** $\displaystyle\int kf(x)\,dx = k\int f(x)\,dx$ for a constant c.

Let's look at few examples involving these indefinite integrals.

Example 1.4. Evaluate

$$\int 4e^{2-3x}\,dx$$

Solution. By the indefinite integrals discussed above,

$$\int 4e^{2-3x}\,dx = 4e^2\int e^{-3x}\,dx = -\frac{4e^2}{3}e^{-3x} + C.$$

□

Example 1.5. Evaluate

$$\int \frac{x^3-2x+3}{x^2}\,dx$$

Solution. By the indefinite integrals discussed above,

$$\begin{aligned}
\int \frac{x^3-2x+3}{x^2}\,dx &= \int x\,dx - \int \frac{2}{x}\,dx + \int \frac{3}{x^2}\,dx \\
&= \frac{x^2}{2} - 2\int \frac{1}{x}\,dx + 3\int x^{-2}\,dx \text{ (By power and multiples rule)} \\
&= \frac{x^2}{2} - 2\ln|x| - \frac{3}{x} + C.
\end{aligned}$$

□

1.1.4. Initial Value Problem

We can think of an antiderivative as a solution to the **differential equation**

$$\frac{dy}{dx} = f(x)$$

We won't go too deep into differential equations in this course. There is a separate course for that. But in short, a differential equation is an equation involving a function and its derivatives. Solving them requires different techniques depending on the equation itself.

For the equation above, we are solving for the function y, i.e. we are trying to find a function $y = F(x)$ whose derivative is $f(x)$. As we have seen before there are several functions which satisfies that (since $f(x)$ has several antiderivatives). However, if we are given an **initial condition** (a given value of y for a specific value of x) then we can find a **particular solution** to the above equation. Let's look into it through an example.

Example 1.6. Solve the initial value problem.

$$\frac{dy}{dx} = x^3, \qquad y(0) = 4.$$

Solution. The general antiderivative is,

$$y(x) = \int x^3\, dx = \frac{1}{4}x^4 + C. \text{ (by the Power rule)}$$

To solve for C we use the initial condition that is given to us. We substitute the values of x and y and then solve for C.

$$\begin{aligned} y(0) &= \frac{1}{4}0^4 + C. \\ 4 &= C. \end{aligned}$$

Therefore, our solution is $y(x) = \frac{1}{4}x^4 + 4$.

□

1.2. Approximating and Computing area

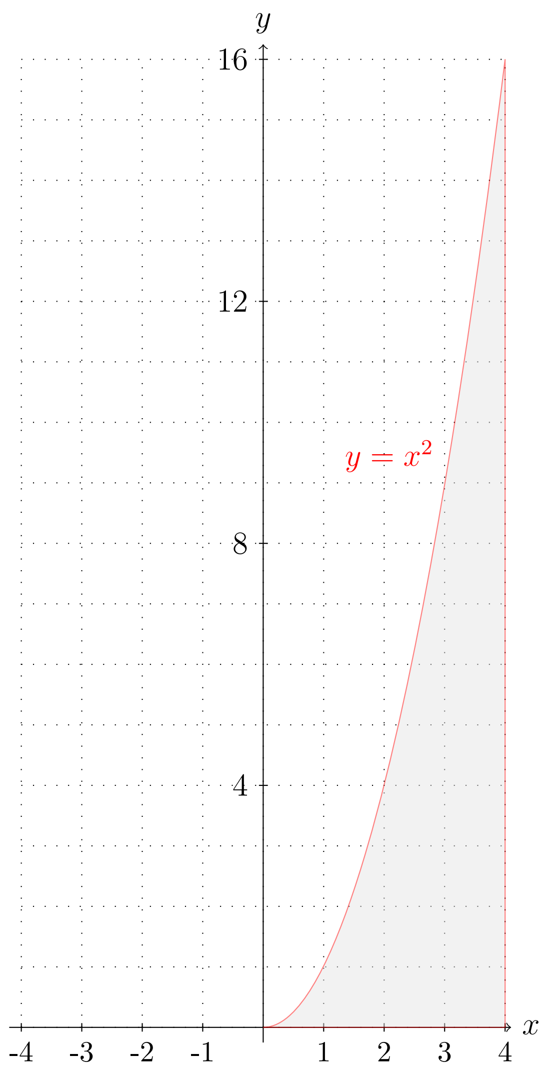
1.2.1. Performance Criteria

(a) Approximate a definite integral using a finite sum of areas of rectangles.
(b) Use a graph to determine the value of a definite integral.

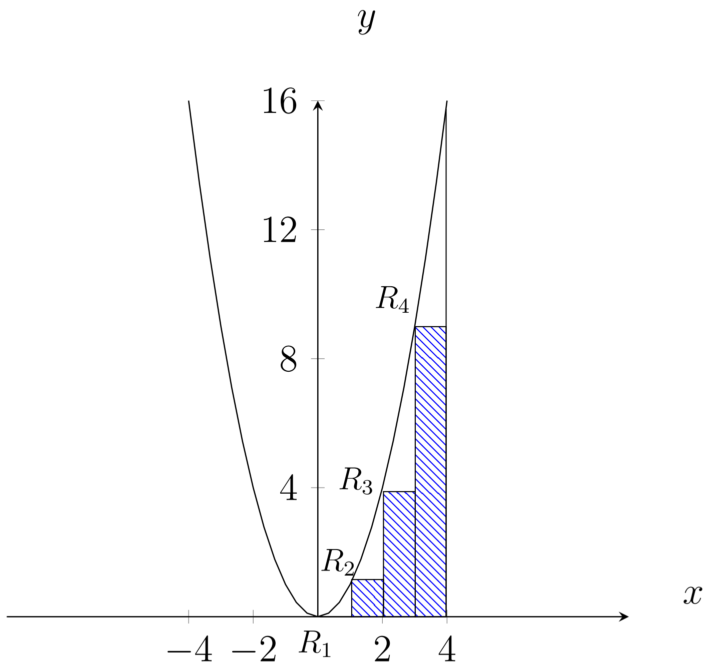
(c) Express a definite integral as a limit of sums or vice-versa.
(d) Compute a definite integral using a limit of sums.

1.2.2. Approximating area by Rectangles

In this section we will try to approximate the area under a curve using rectangles. Consider the graph of the simple parabola $f(x) = x^2$. Let's say we want to approximate the area under this curve on the interval $[0, 4]$.



As of now, we don't have any specific method which we can use to find the area of this shaded region. However, we can always approximate it by means of rectangles. Look at the figure below.



There are 3 rectangles that we can clearly see. However, in reality there are 4 rectangles. There is a rectangle having $(0, 0)$ and $(1, 0)$ as opposite vertices. We cannot seem to see it since it's meshed onto the x axis.

Another thing to note here is that the top left vertex of each rectangle touches the graph of $y = x^2$.

Approximating a region in this way is referred to as approximating using the **left end points**.

Let us find the area of each rectangle separately. To calculate the area we need to know the width and length of each rectangle. The width of each rectangle is $\Delta x = \dfrac{(4-0)}{4} = 1$, whereas the length depends upon the **left end point** of each interval on which the rectangle lies.

- Area of R_1 = Width of $R_1\times$ Length of $R_1 = 1\times f(0) = 1\times 0 = 0$.
- Area of R_2 = Width of $R_2\times$ Length of $R_2 = 1\times f(1) = 1\times 1 = 1$.
- Area of R_3 = Width of $R_3\times$ Length of $R_3 = 1\times f(2) = 1\times 4 = 4$.
- Area of R_4 = Width of $R_4\times$ Length of $R_4 = 1\times f(3) = 1\times 9 = 9$.
- Total Area $= 0 + 1 + 4 + 9 = 14$.

In short we can say that $L_4 = 14$. The L stands for **left end point** and 4 signifies the number of rectangles used.

Is this answer close to the exact value? The exact area of the shaded region under the parabola from the first diagram is $\dfrac{64}{3} = 21.33$

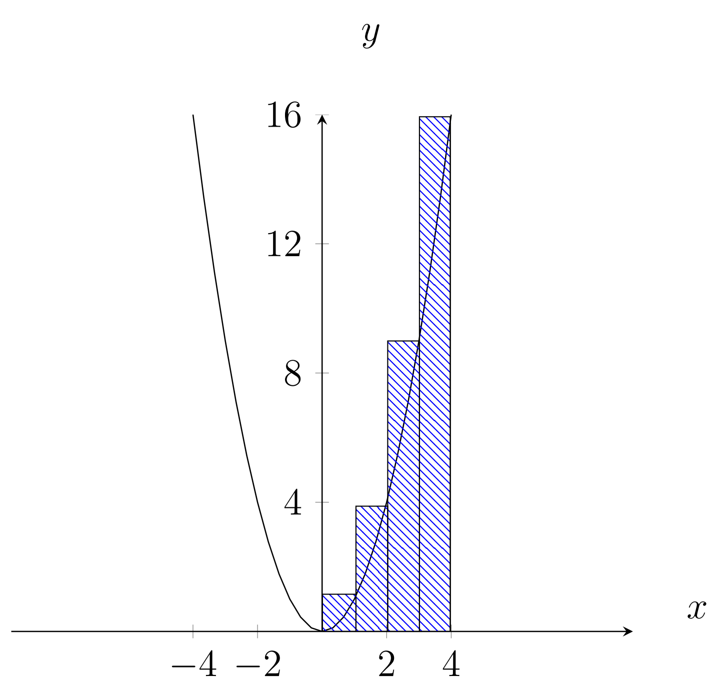
(How? Well, for the moment just believe me). So, there is quite a bit of difference between the approximate area that we found out and what the exact area is. A natural question at this point would be to find ways to better the approximation. One way to better the approximation would be to use more number of rectangles.

Additional Problem. When we computed L_4, do you see why L_4 was less than the exact area?

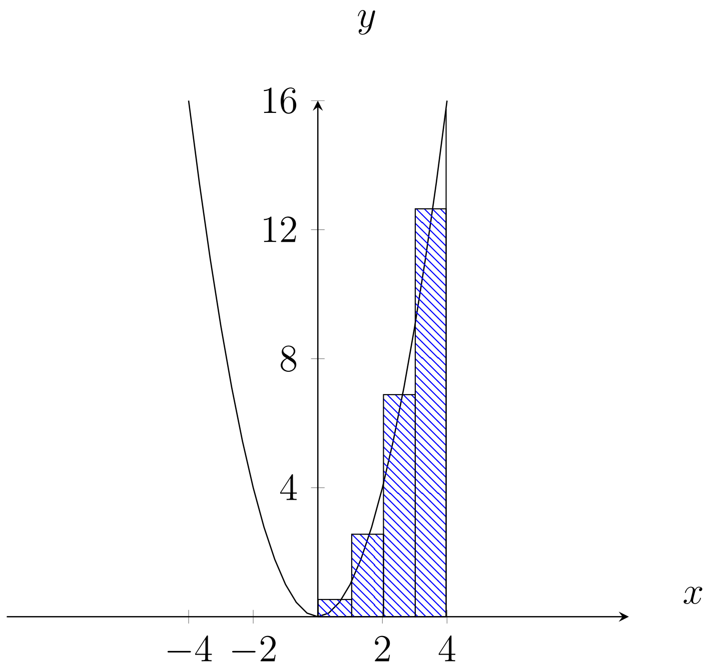
For the function $f(x) = x^2$, compute L_8 and L_{16}. Compare these values with the exact area.

Once you do the above problem, you will see that L_{16} gives a better approximation than L_8 (i.e. L_{16} is closer to 21.33 than L_8).

1.2.2.1. **Right end points and Midpoints.** The above calculations can also be done by picking the **right end point** (see below)



and **mid point** (see below) of every interval on which the rectangle lies.



Additional Problem. Compute R_4 (Area using right end points) and M_4 (Area using mid points) for the graph of $y = x^2$ on $[0, 4]$. Compare these results with L_4 and the exact area.

1.2.3. A general approach using limits

As I mentioned before, we can get a better approximation of the area under the curve if we use more number of rectangles. Let us try to find a general expression of this area. Let us choose the left end points of each interval for our calculation.

Example 1.7. Compute the area L_N, using N rectangles under the graph of the curve $y = f(x)$ on the interval $[a, b]$.

Solution. Width of each interval (and rectangle) $= \dfrac{\text{Total width of the interval}}{\text{Number of rectangles}}$

$$= \frac{b-a}{N} = \Delta x.$$

Length of the kth rectangle is $f(a + (k-1)\Delta x)$ (Think about it). Therefore, total area of the region is the sum of the areas of all the rectangles which is

$$
\begin{aligned}
L_N &= \Delta x \times f(a) + \Delta x \times f(a+\Delta x) + \cdots + \Delta x \times f(a+(N-1)\Delta x) \\
&= \Delta x(f(a) + f(a+\Delta x) + \cdots + f(a+(N-1)\Delta x)) \\
&= \Delta x \sum_{j=0}^{N-1} f(a + j\Delta x)
\end{aligned}
$$

$\square$

Additional Problem. Compute R_N and M_N for the graph of $y = f(x)$ on the interval $[a, b]$ using N rectangles.

However many rectangles we pick, the above method will still give us an approximation of the area (albeit a better one, the more number of rectangles we choose). Is there any way to compute the exact area using this method?

Yes, if we take *infinite* number of rectangles.

Example 1.8. Compute the area under the curve $y = x^2$ on the interval $[0, 4]$ using *infinite* number of rectangles.

Solution. Let us use left end points for the intervals and let's start with N rectangles.

Width of each interval $= \Delta x = \frac{4-0}{N} = \frac{4}{N}$.

By using the previous formula,

$$
\begin{aligned}
L_N = \Delta x \sum_{j=0}^{N-1} f(a + j\Delta x) &= \frac{4}{N} \sum_{j=0}^{N-1} \sum_{j=0}^{N-1} f(0 + j\frac{4}{N}) \\
&= \frac{4}{N} f(j\frac{4}{N}) \\
&= \frac{4}{N} \sum_{j=0}^{N-1} \frac{16j^2}{N^2} \\
&= \frac{64}{N^3} \sum_{j=0}^{N-1} j^2
\end{aligned}
$$

We can pull 16 and N^2 out since they are constants

So what is $\sum_{j=0}^{N-1} j^2$?

$$
\sum_{j=0}^{N-1} j^2 = 0^2 + 1^2 + \cdots + (N-1)^2 = \frac{(N-1)N(2N-1)}{6}
$$

Therfore,

$$L_N = \frac{64}{N^3}\sum_{j=0}^{N-1} j^2 = \frac{64}{N^3}\cdot\frac{(N-1)N(2N-1)}{6} = \frac{32(N-1)N(2N-1)}{3N^3}$$

If we substitute $N = 4$, we end up getting $L_4 = 14$, which is what we had before. So how do we use infinite number of rectangles? We could try to find the limit of L_N as $N \to \infty$. Hence,

$$\begin{aligned}
\text{Area} = \lim_{N\to\infty} L_N &= \lim_{N\to\infty} \frac{32(N-1)N(2N-1)}{3N^3} \\
&= \lim_{N\to\infty} \frac{32(1-\frac{1}{N})(\frac{N}{N})(2-\frac{1}{N})}{3} \\
&= \frac{64}{3} \\
&= 21.33 \text{ (Remember when I said earlier to believe me).}
\end{aligned}$$

□

Additional Problem. Do the above problem by using right end points and mid points for the interval. Do you end up with the same answer as above?

Once you do the above additional problem you will see that we could choose any of the three points (left end, right end and mid) from our interval and all of them give the same area under the curve. In general, they all approach the same limit

$$\lim_{N\to\infty} R_N = \lim_{N\to\infty} L_N = \lim_{N\to\infty} M_N = \text{Area under the curve.}$$

Handout 2
MATH 252

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2.1. The Definite Integral

2.1.1. Performance Criteria

(a) Use a graph to determine the value of a definite integral.
(b) Express a definite integral as a limit of sums or vice-versa.
(c) Use the properties of definite integrals in a problem.

2.1.2. Riemann Sums

In the previous section, we discussed ways in which we can approximate the area under the curve of a function $f(x)$. We used the left end, right end and mid points as our sample points. By picking an infinite number of rectangles on the interval $[a, b]$ we transformed the approximation into the exact area, i.e.,

$$\lim_{N\to\infty} L_N = \lim_{N\to\infty} R_N = \lim_{N\to\infty} M_N = L \text{ (The exact area)}$$

We call L the **definite integral** of $f(x)$ over $[a, b]$. Before we venture more into definite integrals, let us look into a generalization of the method that we used earlier using rectangles called **Riemann sums**.

When computing Riemann sums, we can relax certain requirements that we had previously.

- **Partition** - The rectangles need not have equal width. We can divide the interval $[a, b]$, with points $x_0, x_1, \ldots, x_N$, such that

$$P : a = x_0 < x_1 < x_2 < \cdots < x_N = b$$

  where P is a partition of size N.
- **Sample points** - We can pick sample points

$$C = \{c_1, \ldots, c_N\}$$

such that c_i belongs to the subinterval $[x_{i-1}, x_i]$ for all i. We don't have to be restricted to just the end points and mid point.

Since we can relax the width of each rectangle, therefore the length of each subinterval is different. The width of the ith subinterval $[x_{i-1}, x_i]$ is

$$\Delta x_i = x_i - x_{i-1}$$

and the length of the ith rectangle is $f(c_i)$. The **norm** of P, denoted by $||P||$ is the maximum of the widths Δx_i. Hence, area of the ith rectangle is given by

$$A_i = f(c_i)\Delta x_i$$

Thus the Riemann sum is the sum

$$R(f,P,C) = \sum_{i=1}^{N} A_i = \sum_{i=i}^{N} f(c_i)\Delta x_i = f(c_1)\Delta x_1 + \cdots + f(c_N)\Delta x_N$$

If for a specific i, $f(c_i) \geq 0$, then the rectangle is above the x-axis, whereas if $f(c_i) < 0$, then the rectangle extends below the x-axis, and then $f(c_i)\Delta x_i$ is the negative of the area.

Example 2.1. Calculate the Riemann sum $R(f,P,C)$ for the given function, partition, and choice of sample points.

$$f(x) = 2x + 3, \quad P = \{-4, -1, 1, 4, 8\}, \quad C = \{-3, 0, 2, 5\}$$

Solution. Here $c_1 = -3, c_2 = 0, c_3 = 2, c_4 = 5$ and $x_0 = -4, x_1 = -1, x_2 = 1, x_3 = 4, x_4 = 8$. Therefore,
$f(c_1) = f(-3) = -3$.
$f(c_2) = f(0) = 3$.
$f(c_3) = f(2) = 7$.
$f(c_4) = f(5) = 13$.
$\Delta x_1 = x_1 - x_0 = 3$.
$\Delta x_2 = x_2 - x_1 = 2$.
$\Delta x_3 = x_3 - x_2 = 3$.
$\Delta x_4 = x_4 - x_3 = 4$.
Hence, the areas of the rectangles are
$A_1 = f(c_1)\Delta x_1 = -9$.
$A_2 = f(c_2)\Delta x_2 = 6$.
$A_3 = f(c_3)\Delta x_3 = 21$.
$A_4 = f(c_4)\Delta x_4 = 52$.
Therefore,

$$R(f,P,C) = A_1 + A_2 + A_3 + A_4 = -9 + 6 + 21 + 52 = 70.$$

□

As before, the more number of rectangles we pick, the better approximation we have, or in the context of this section, as the norm $||P||$ tends to zero, the approximations get better.

Definition 2.2. The **definite integral** of $f(x)$ over $[a, b]$, denoted by the integral sign, is limit of Riemann sums

$$\int_a^b f(x)\,dx = \lim_{||P||\to 0} R(f, P, C) = \lim_{||P||\to 0} \sum_{i=i}^{N} f(c_i)\Delta x_i$$

When this limit exists, we say that $f(x)$ is integrable over $[a, b]$.

The endpoints a and b of $[a, b]$ are called the **limits of integration**.

2.1.3. Signed Area

When $f(x) \geq 0$, the definite integral represents the area under the graph of $y = f(x)$. When $f(x)$ takes on both positive and negative values then we can define the notion of **signed area**.

Signed area of a region = (Area above the x-axis)-(Area below the x-axis)

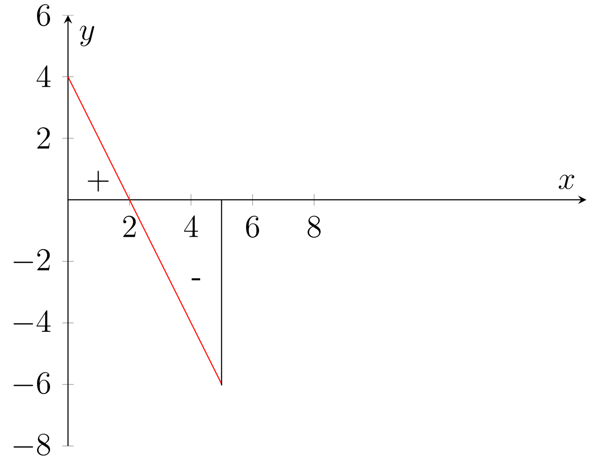
The definite integral that we have defined previously represents the signed area of the region between the graph and the x-axis, i.e.,

$$\int_a^b f(x)\,dx = \text{Signed area of the region between the graph and } x \text{ axis over } [a, b]$$

Example 2.3. Calculate

$$\int_0^5 (4 - 2x)\,dx$$

Solution. Let us draw the graph of the function $y = 4 - 2x$ on $[0, 5]$.



There are two triangles here, one above the x-axis whose area is $\frac{1}{2} \times 2 \times 4 = 4$ and another below the x-axis whose area is $\frac{1}{2} \times 3 \times 6 = 9$. Therefore the signed area of the entire region is $4 - 9 = -5$. Hence,

$$\int_0^5 (4 - 2x)\, dx = -5$$

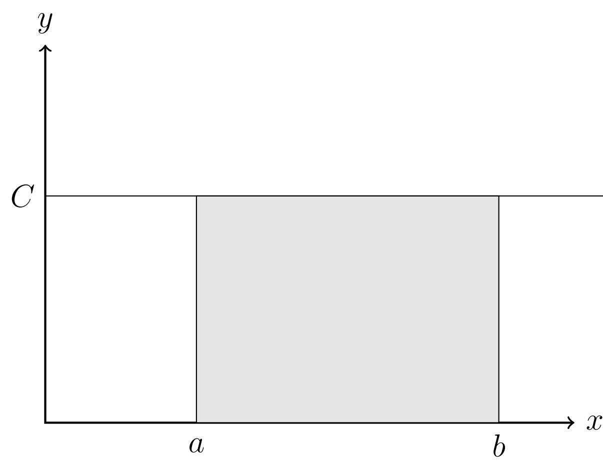
□

2.1.4. Properties of the Definite Integral

- **Integral of a Constant** - For any constant C,

$$\int_a^b C\, dx = C(b - a)$$

  The function here is $y = C$ which is a horizontal straight line passing through C. Hence we are looking at finding the area of the shaded region below.



The width of the rectangle is $b-a$ and the height is C. Hence the area of the shaded region is $C(b-a)$.

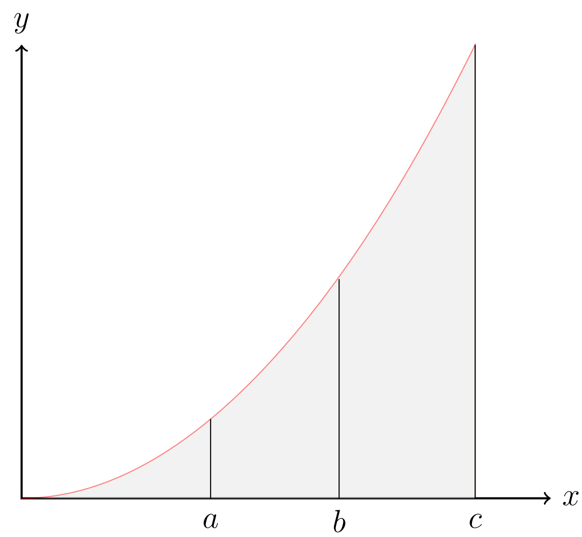
- **Linearity of the Definite Integral** - If f and g are integrable over $[a,b]$, then $f+g$ and Cf are integrable (for any constant C), and
  (a) $\int_a^b (f(x)+g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$
  (b) $\int_a^b Cf(x)\,dx = C\int_a^b f(x)\,dx$
- **Reversing the Limits** - For $a<b$, we have

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

- **Additivity for Adjacent Intervals** - Let $a \le b \le c$, and assume that $f(x)$ is integrable. Then

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$$

This can be seen by noting that the area over $[a,c]$ is the **sum** of the areas over $[a,b]$ and $[b,c]$ in the diagram below.



Example 2.4. Using the fact that $\int_0^a x^2\,dx = a^3/3$, calculate

$$\int_3^8 x^2\,dx$$

Solution. By the additive property above,

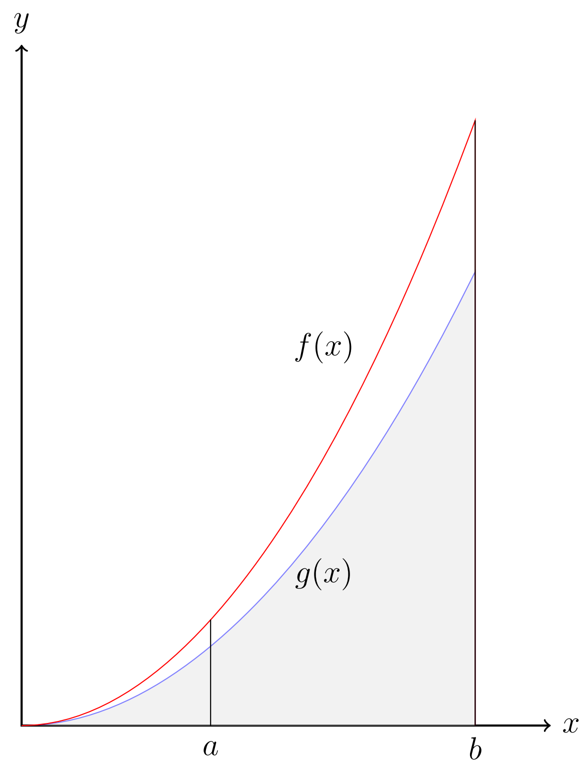
$$\begin{aligned}\int_0^3 x^2\,dx + \int_3^8 x^2\,dx &= \int_0^8 x^2\,dx \\ \int_3^8 x^2\,dx &= \int_0^8 x^2\,dx - \int_0^3 x^2\,dx \\ &= \frac{8^3}{3} - \frac{3^3}{3} = \frac{512}{3} - \frac{27}{3} = \frac{485}{3}\end{aligned}$$

$\square$

- **Comparison Theorem** - If f and g are integrable and $g(x) \leq f(x)$ for x in $[a,b]$, then

$$\int_a^b g(x)\,dx \leq \int_a^b f(x)\,dx$$

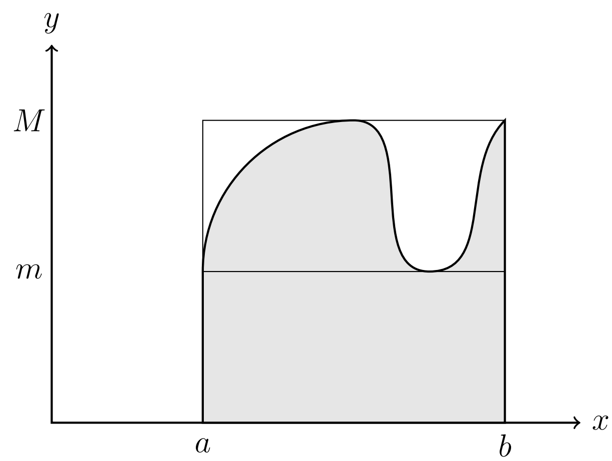
It is clear from the diagram below that, the area of the region below $f(x)$ on $[a,b]$ is greater than the area of the region below $g(x)$.



- **Extremum theorem** - Suppose that there are numbers m (lower bound) and M (upper bound) such that $m \leq f(x) \leq M$ for x in $[a,b]$. Then

$$\boxed{m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)}$$

It is clear from the figure below that the area of the shaded region which is $\int_a^b f(x)\,dx$ lies between the areas of the rectangle with height m and M. Note that the area of the rectangle with height m is $m(b-a)$ and area of the rectangle with height M is $M(b-a)$.



Example 2.5. Prove that

$$\frac{1}{3} \leq \int_4^6 \frac{1}{x}\,dx \leq \frac{1}{2}$$

Solution. $f(x) = \dfrac{1}{x}$ is a decreasing function on $(0, \infty)$. Therefore, on the interval $[4, 6]$, its maximum value (M) is $\dfrac{1}{4}$ and its minimum value (m) is $\dfrac{1}{6}$. By the Extremum theorem from above,

$$\frac{1}{6} \cdot (6-4) \leq \int_4^6 \frac{1}{x}\,dx \leq \frac{1}{4} \cdot (6-4)$$

$$\frac{1}{3} \leq \int_4^6 \frac{1}{x}\,dx \leq \frac{1}{2}$$

□

2.2. The Fundamental Theorem of Calculus, Part I

2.2.1. Performance Criteria

(a) Use the Fundamental Theorem of Calculus to differentiate an integral of the form $\displaystyle\int_a^x f(t)\,dt$.

(b) Use the Fundamental Theorem of Calculus to evaluate a definite integral.

2.2.2. The Fundamental theorem of calculus, part I

As we have seen before that if $F(x)$ is an antiderivate of $f(x)$, then $\int f(x)\,dx = F(x) + C$, we now have a similar result for definite integrals.

Theorem 2.6. *Assume that $f(x)$ is continuous on $[a,b]$. If $F(x)$ is an antiderivative of $f(x)$ on $[a,b]$, then*

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

$F(b) - F(a)$ is denoted by $F(x)\Big|_a^b$. With this notation, the Fundamental theorem of calculus reads,

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b$$

Example 2.7. Evaluate

$$\int_0^{\pi/4} \sec^2 x\,dx$$

Solution. Recall that $\dfrac{d}{dx}(\tan x) = \sec^2 x$. Therefore,

$$\int_0^{\pi/4} \sec^2 x\,dx = \tan x\Big|_0^{\pi/4} = \tan(\frac{\pi}{4}) - \tan(0) = 1 - 0 = 1.$$

□

Example 2.8. Evaluate

$$\int_0^1 \frac{1}{t+1}\,dt$$

Solution. Recall that $\dfrac{d}{dt}\ln|t+1| = \dfrac{1}{t+1}$. Hence,

$$\int_0^1 \frac{1}{t+1}\,dt = \ln|t+1|\Big|_0^1 = \ln|1+1| - \ln|1+0| = \ln 2.$$

□

Handout 4
MATH 252

Dibyajyoti Deb

4.1. Further Transcendental Functions

4.1.1. Performance Criteria

(a) Use substitution on integrals involving natural logarithm and other functions.

4.1.2. Some more standard integrals

In this section we look at some common functions that we have already seen before and see how they can be written as the integral of some other functions. We first look at the the natural logarithm function.

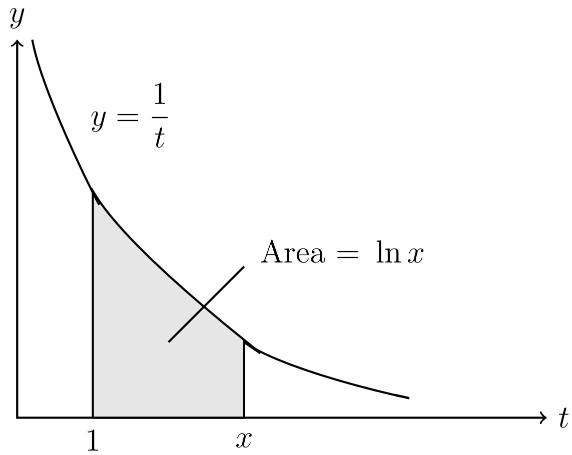
4.1.2.1. **Natural Logarithm.** From differential calculus we know that,

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

Hence, by the Fundamental Theorem of Calculus Part I we have

$$\ln x = \int_1^x \frac{1}{t}\,dt \qquad \text{for } x > 0$$

Thus, we can define $\ln x$ to be the area under the hyperbola $y = 1/t$ from 1 to x.



The area of the shaded region in the graph above is precisely $\ln x$.

4.1.2.2. **Inverse Trigonometric Functions.** Just like before we can use the derivatives of the inverse trigonometric functions to come up with various standard integrals.

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}, \qquad \int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C$$

$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}, \qquad \int \frac{dx}{1+x^2} = \tan^{-1} x + C$$

$$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}, \qquad \int \frac{dx}{|x|\sqrt{x^2-1}} = \sec^{-1} x + C$$

$$\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}, \qquad \int -\frac{dx}{\sqrt{1-x^2}} = \cos^{-1} x + C$$

$$\frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}, \qquad \int -\frac{dx}{1+x^2} = \cot^{-1} x + C$$

$$\frac{d}{dx}(\csc^{-1} x) = -\frac{1}{|x|\sqrt{x^2-1}}, \qquad \int -\frac{dx}{|x|\sqrt{x^2-1}} = \csc^{-1} x + C$$

We can use these results to evaluate integrals of even more complicated functions.

Example 4.1. Evaluate

$$\int \frac{dx}{x\sqrt{25x^2-1}}$$

Solution. Our goal here is to use one of new standard integrals that we covered in this section. However we would need to have a x^2 instead of the $25x^2$ in our expression before we can use the standard integrals. Hence we simplify the expression a little bit

$$\int \frac{dx}{x\sqrt{25x^2-1}} = \int \frac{dx}{x\sqrt{(5x)^2-1}}$$

We use the substitution

$$\begin{aligned} u &= 5x \\ \text{Therefore, } du &= 5\,dx \\ dx &= \frac{1}{5}\,du \end{aligned}$$

Thus,

$$\begin{aligned} \int \frac{dx}{x\sqrt{(5x)^2-1}} &= \int \frac{\frac{1}{5}du}{\frac{u}{5}\sqrt{u^2-1}} \\ &= \int \frac{du}{u\sqrt{u^2-1}} \\ &= \sec^{-1} u + C \\ &= \sec^{-1}(5x) + C \end{aligned}$$

□

Example 4.2. Evaluate

$$\int_{-\ln 2}^{0} \frac{e^x\,dx}{1+e^{2x}}$$

Solution. Here we notice that the $e^{2x} = (e^x)^2$ and the derivative of e^x (which is also e^x) is present in the expression. Therefore we use the substitution

$$\begin{aligned} u &= e^x \\ du &= e^x\,dx \end{aligned}$$

The new limits are

$$x = -\ln 2 \Rightarrow u = e^{-\ln 2} = 1/2$$

$$x = 0 \Rightarrow u = e^0 = 1$$

Thus,

$$\begin{aligned}
\int_{-\ln 2}^{0} \frac{e^x\,dx}{1+e^{2x}} &= \int_{\frac{1}{2}}^{1} \frac{du}{1+u^2} \\
&= \tan^{-1} u \Big|_{\frac{1}{2}}^{1} \\
&= \tan^{-1}(1) - \tan^{-1}(\frac{1}{2}) \\
&= 0.7854 - 0.4636 = 0.3218.
\end{aligned}$$

□

4.1.2.3. **General exponential function.** The general exponential function is of the form

$$f(x) = a^x, \qquad a > 0, a \neq 1$$

Note, that when a is e we end up with the natural exponential function e^x. From differential calculus we know that

$$\frac{d}{dx}(a^x) = a^x \ln a.$$

Hence,

$$\frac{d}{dx}(\frac{a^x}{\ln a}) = a^x.$$

Therefore,

$$\int a^x\,dx = \frac{a^x}{\ln a} + C$$

Example 4.3. Evaluate

$$\int (\sin x)5^{\cos x}\,dx$$

Solution. The derivative of $\cos x$ is $-\sin x$ and it is present in that form in our expression. Thus,

$$\begin{aligned}
u &= \cos x \\
du &= -\sin x\,dx \\
\sin x\,dx &= -du
\end{aligned}$$

Thus,

$$\begin{aligned}
\int (\sin x)5^{\cos x}\,dx &= \int 5^u\,(-du) \\
&= -\int 5^u\,du \\
&= -\frac{5^u}{\ln 5} + C \\
&= -\frac{5^{\cos x}}{\ln 5} + C
\end{aligned}$$

□

4.2. Area between two curves

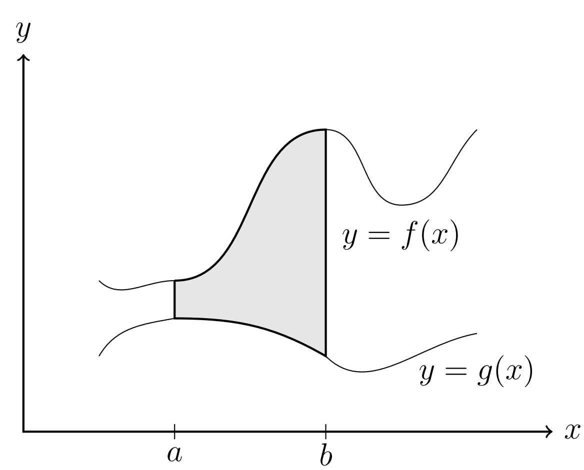
4.2.1. Performance Criteria

(a) Use a definite integral to find the area between two curves.

4.2.2. Area between two curves

In this section we will learn how to find the area between two curves. This is an important application of integration as this would help us to study and interpret different types of graphs.

4.2.2.1. **Integration along the x-axis.** Consider the two curves below.

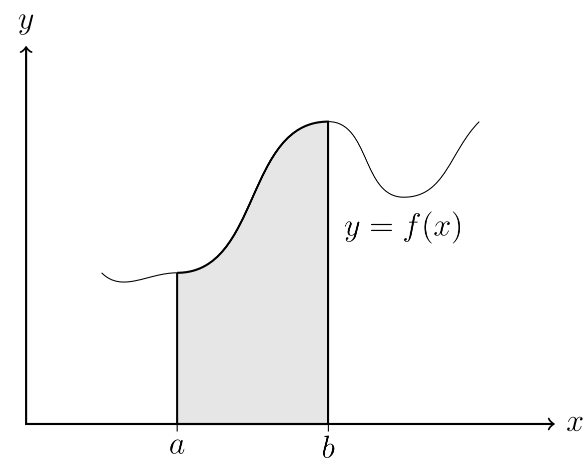


We would like to find the area of the shaded region between the two curves $y = f(x)$ and $y = g(x)$. This can be done by

(1) First finding the integral

$$\int_a^b f(x)\,dx$$

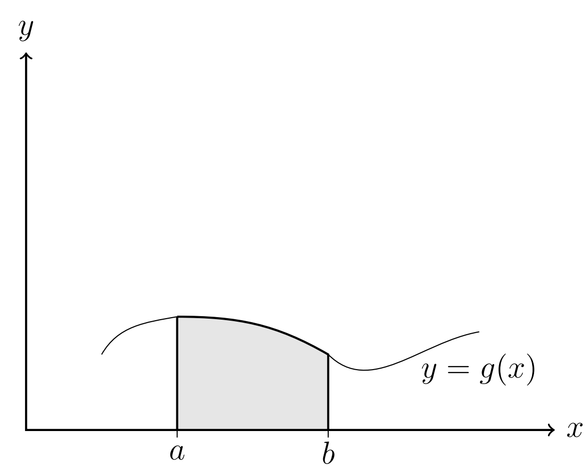
which would give the area of the shaded region below



(2) And then finding the integral

$$\int_a^b g(x)\,dx$$

which would give the area of the shaded region below

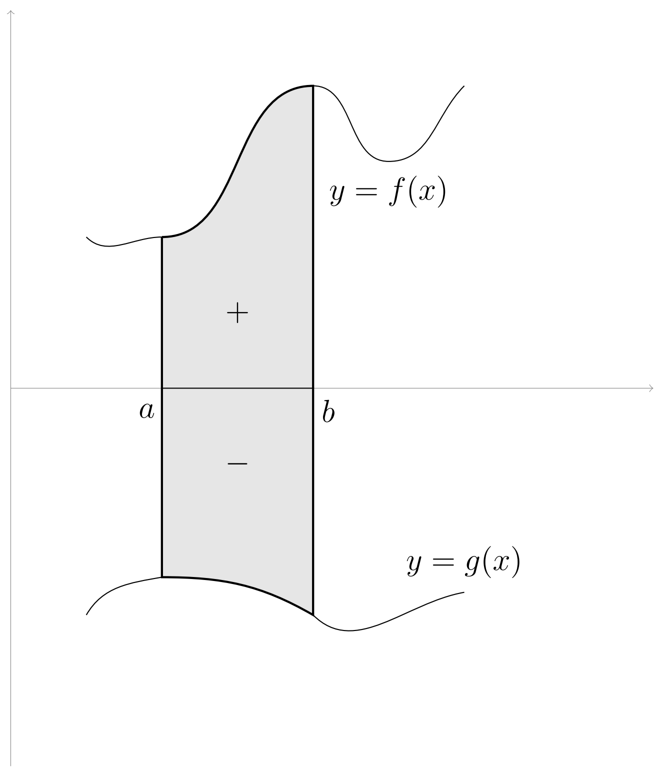


(3) And then subtracting the integral in (2) from the integral in (1).

Thus,

$$\begin{aligned}\text{Area between the graphs} &= \int_a^b f(x)\,dx - \int_a^b g(x)\,dx \\ &= \int_a^b (f(x) - g(x))\,dx\end{aligned}$$

However, what if one or both of the curves lie below the x-axis. Does the integral for computing the area between the curves change in any way?



The area of the above shaded region is the sum of the areas above the x-axis and the area below the x-axis. Now the area above the x-axis is given by the integral

$$\int_a^b f(x)\,dx$$

which is a positive value since the region lies above the x-axis. The area below the x-axis is given by

$$-\int_a^b g(x)\,dx$$

Note the negative sign as $\int_a^b g(x)\,dx$ is the signed area of the region below the x-axis, therefore in order to make it positive we have to multiply by -1. Hence,

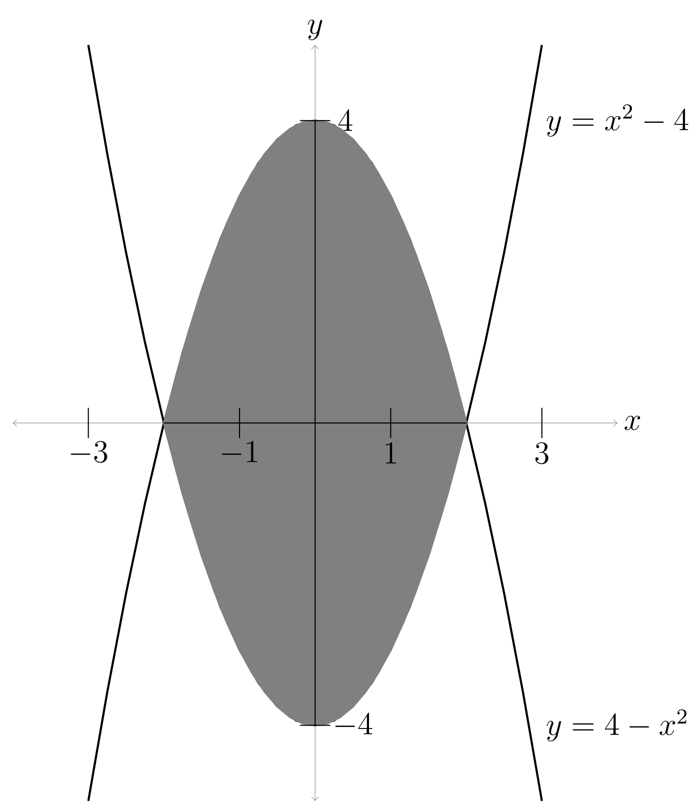
$$\begin{aligned}\text{Area between the graphs} &= \int_a^b f(x)\,dx + (-\int_a^b g(x))\,dx \\ &= \int_a^b (f(x)-g(x))\,dx\end{aligned}$$

Thus it seems that no matter where the curves lie, if $f(x) \geq g(x)$ on $[a,b]$ then,

$$\begin{aligned}\text{The area between the curves} &= \int_a^b (f(x)-g(x))\,dx \\ &= \int_a^b (y_{\text{Curve on top}} - y_{\text{Curve on bottom}})\,dx\end{aligned}$$

Example 4.4. Find the area of the region enclosed by the curves $y = 4 - x^2$ and $y = x^2 - 4$.

Solution. Let us first sketch the two curves.



The points of intersection of these two curves can be found by

$$\begin{aligned} x^2 - 4 &= 4 - x^2 \\ 2x^2 &= 8 \\ x^2 &= 4 \\ x &= \pm 2 \end{aligned}$$

On the interval $[-2, 2]$ the curve on top is $y = 4 - x^2$ and the curve at the bottom is $y = x^2 - 4$. Hence,

$$\begin{aligned} \text{Area of the shaded region} &= \int_{-2}^{2} (4 - x^2 - (x^2 - 4))\, dx \\ &= \int_{-2}^{2} (8 - 2x^2)\, dx \\ &= (8x - \frac{2x^3}{3})\Big|_{-2}^{2} \\ &= \frac{64}{3} \end{aligned}$$

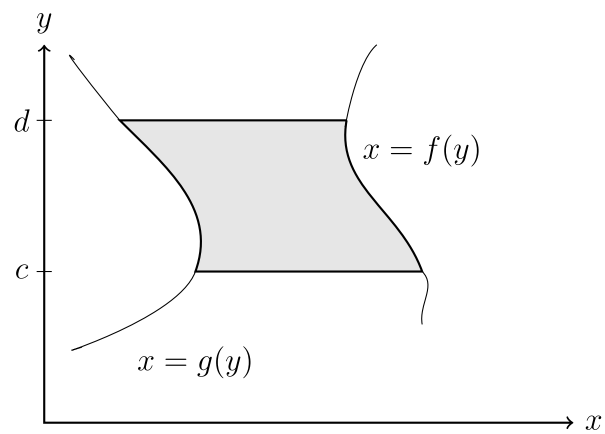
□

4.2.2.2. **Integration along the y-axis.** Suppose that we are given x as function of y, i.e. $x = f(y)$. What does it mean to evaluate the integral

$$\int_c^d f(y)\, dy$$

Just like our previous definite integrals, this represents the signed area, where regions to the right of the y-axis have positive area and regions to the left of the y-axis have negative area. Thus,

$$\int_c^d g(y)\, dy = \text{ signed area between the graph and the } y\text{−axis for } c \leq y \leq d$$



Just like in the previous section,

$$\begin{aligned}\text{The area between the curves} &= \int_c^d (f(y) - g(y))\, dy \\ &= \int_c^d (x_{\text{Curve on right}} - x_{\text{Curve on left}})\, dy\end{aligned}$$

Handout 3
MATH 252

Dibyajyoti Deb

3.1. The Fundamental Theorem of Calculus, Part II

3.1.1. Performance Criteria

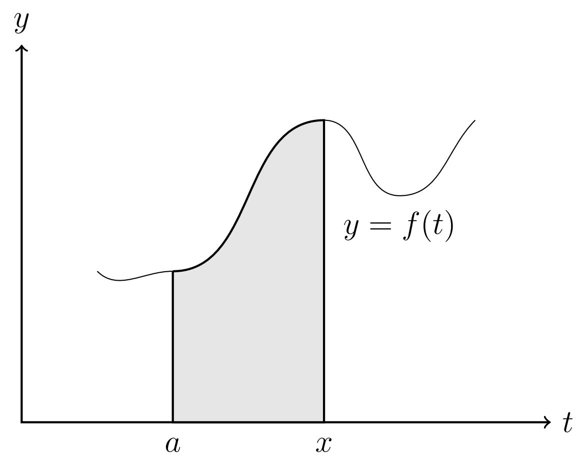
(a) Use the Fundamental Theorem of Calculus to differentiate an integral of the form $\int_a^x f(t)\,dt$.

(b) Use the Fundamental Theorem of Calculus to evaluate a definite integral.

3.1.2. The Fundamental theorem of Calculus, Part II

In this section we look at Part 2 of the Fundamental theorem of calculus. Part I of the Fundamental theorem said that we could use the antiderivative of a function to compute the definite integral. Part 2 on the other hand turns this statement around. It tells us that we can use the definite integral to *construct* the antiderivative.

In order to state Part 2 of the Fundamental theorem, we first need to introduce the **area function** of f.



The area of the shaded region above is a function of x, as we can "move" the x while keeping a constant, across the t-axis which will result in different areas for different values of x. Therefore,

$$A(x) = \int_a^x f(t)\,dt = \text{ Signed area from } a \text{ to } x.$$

Let us look at an example.

Example 3.1. Find a formula for the area function of $f(x) = 2x + 6$ with lower limit $a = 0$.

Solution. The area function of $f(x) = 2x + 6$ is

$$A(x) = \int_0^x (2t + 6)\,dt$$

Note that we replaced the x with t in our function. Now the function $F(t) = t^2 + 6t$ is an antiderivative of $f(t) = 2t + 6$. Therefore, by Part I of the Fundamental Theorem of Calculus we have

$$A(x) = \int_0^x (2t+6)\,dt = F(x) - F(0) = (x^2+6x) - (0^2+6\cdot 0) = x^2+6x$$

Hence,

$$A(x) = x^2 + 6x$$

$\square$

Notice that in the previous example, $A'(x) = f(x)$. This is true in general which we summarize in this next theorem.

Theorem 3.2. *Assume that $f(x)$ is continuous on an open interval I and let $a \in I$. Then the area function*

$$A(x) = \int_a^x f(t)\,dt.$$

is an antiderivative of $f(x)$ on I i.e. $A'(x) = f(x)$. Equivalently,

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x).$$

The above theorem is generally known as the **Fundamental Theorem of Calculus, Part II**. We will be mostly using this theorem when solving problems, however, for the sake of generality, I also present the more general case of the same theorem.

Theorem 3.3. *Assume that $f(x)$ is continuous on an open interval I containing the differentiable function $y = g(x)$ and $y = h(x)$, then*

$$\frac{d}{dx}\int_{g(x)}^{h(x)} f(t)\,dt = f(h(x))h'(x) - f(g(x))g'(x).$$

Proof. The proof of the above theorem uses the first part of the fundamental theorem. By Part I, if $F(x)$ is an antiderivative of $f(x)$ i.e. $F'(x) = f(x)$, then

$$\int_{g(x)}^{h(x)} f(t)\,dt = F(h(x)) - F(g(x))$$

Therefore, differentiating both sides with respect to x we have,

$$\begin{aligned}
\frac{d}{dx}\int_{g(x)}^{h(x)} f(t)\,dt &= \frac{d}{dx}F(h(x)) - \frac{d}{dx}F(g(x)) \\
&= F'(h(x)) \cdot h'(x) - F'(g(x)) \cdot g'(x) \\
&= f(h(x))h'(x) - f(g(x))g'(x) \text{ (Since } F'(x) = f(x))
\end{aligned}$$

□

Theorem 3.2 is a special case of Theorem 3.3. This can be seen by taking $h(x) = x$ and $g(x) = a$.

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x) \cdot 1 - f(a) \cdot 0 = f(x).$$

Now, let us look at some examples.

Example 3.4. Calculate the derivative.

$$\frac{d}{dt}\int_{100}^{t} \sec(5x-9)\,dx$$

Solution. Here $f(x) = \sec(5x-9)$, therefore, by Theorem 3.2,

$$\frac{d}{dt}\int_{100}^{t} \sec(5x-9)\,dx = f(t) = \sec(5t-9)$$

$\square$

Example 3.5. Calculate the derivative.

$$\frac{d}{dx}\int_{\sqrt{x}}^{x^2} \tan t\,dt$$

Solution. Here $f(t) = \tan t$, $h(x) = x^2$ and $g(x) = \sqrt{x}$. Therefore, by Theorem 3.3, we have

$$\begin{aligned}\frac{d}{dx}\int_{\sqrt{x}}^{x^2} \tan t\,dt &= f(x^2)\cdot 2x - f(\sqrt{x})\cdot\frac{1}{2\sqrt{x}}\\ &= \tan(x^2)\cdot 2x - \tan(\sqrt{x})\cdot\frac{1}{2\sqrt{x}}\end{aligned}$$

$\square$

3.2. Substitution Method

3.2.1. Performance Criteria

(a) Compute an anti-derivative using u-substitution.
(b) Compute an anti-derivative requiring one substitution with a trigonometric identity.
(c) Use u-substitution to change the variable of integration in a definite integral, including changing the limits of integration.

3.2.2. Substitution using Differentials

In this section we finally look at techniques by which we can integrate more complicated functions. Integrating a function in general is harder than differentiating it. There are techniques in differentiation such as the Product rule, Quotient rule and Chain rule which help us in differentiating almost all functions, however that is not the case with integration.

We first look at the **Substitution Method** which acts as the "Chain Rule" for integration. Let us look at this problem,

$$\int (2x+5)^{100}\,dx$$

We could try to foil this expression and then apply the power rule to each term separately however, that would be very tedious as it would involve 101 terms.

Our goal here is somehow change the expression $(2x+5)^{100}$ into an expression that we will be able to integrate using methods that we know such as the Power rule.

Therefore let us try to "collapse" the expression $(2x+5)$ by using a substitution so that we can use the power rule for integration. Let,

$$u = 2x+5$$

Therefore, the problem now becomes

$$\int u^{100}\,dx \tag{3.6}$$

But we have a problem here. We cannot integrate u^{100} which is a function of u with respect to another variable (in this case x due to the presence of dx). So, we need to change from dx to du. How do we do that?

Here, we remind ourselves of the property of differentials. If $u = f(x)$, then $du = f'(x)dx$. Applying this result to our substitution we have,

$$\begin{aligned} du &= 2dx \\ dx &= \frac{1}{2}du \end{aligned}$$

Using this in Equation 3.6 now gives us

$$\int u^{100}\,\frac{1}{2}du = \frac{1}{2}\int u^{100}\,du$$

Finally, we have "replaced" the original expression with an equivalent function by using substitution. Now we can use the power rule for integration.

$$\frac{1}{2}\int u^{100}\,du = \frac{1}{2}\cdot\frac{u^{101}}{101} + C = \frac{u^{101}}{202} + C.$$

We are not done yet. The above expression is not the final answer. Remember that u is a variable that **we** introduced. The original problem was a function of x. Therefore, we need to change our final answer and write it as a function of x. We just replace u with $2x+5$ in our answer above.

$$\int (2x+5)^{100}\,dx = \frac{(2x+5)^{101}}{202} + C.$$

□

Let us look at another problem,

Example 3.7. Evaluate the indefinite integral.

$$\int x^2 \cos(x^3+2)\,dx$$

Solution. As before, the goal here is to reduce the expression using substitution into something that we can easily integrate. The key thing to note here is that the derivative of x^3+2 is $3x^2$ which is already present in our integral (forget about the 3 for a moment). We use the substitution

$$u = x^3 + 2$$

Taking differentials on both sides gives,

$$\begin{aligned} du &= 3x^2 dx \\ x^2 dx &= \frac{1}{3} du \end{aligned}$$

Let us now substitute these new expressions back into the original integral.

$$\begin{aligned} \int x^2 \cos(x^3+2)\,dx &= \int (\cos u) \frac{1}{3}\,du \text{ (Since } x^2 dx = \frac{1}{3} du) \\ &= \frac{1}{3} \int \cos u\,du \\ &= \frac{1}{3} \sin u + C \end{aligned}$$

We finally replace u with x^3+2 and we have our final answer.

$$\int x^2 \cos(x^3+2)\,dx = \frac{1}{3}\sin(x^3+2) + C.$$

□

In both of these problems, the goal was to change the original integrand (the expression that we are integrating) into something that we can integrate easily. Hence, the key was to look for a term in the integrand whose derivative is also present in some form in the integrand itself. Then, we can use that term for our substitution. This happened with $2x+5$ in the first example and x^3+2 in the second example. The derivatives of $2x+5$ and x^3+2 are 2 and $3x^2$ which were already present in the integrand (2 is a constant so it's okay if it isn't present there as it won't affect the substitution).

Let us look at another example.

Example 3.8. Evaluate the indefinite integral.

$$\int \frac{\tan(\ln x)}{x}\, dx$$

Solution. Here, we see that we have $\frac{1}{x}$ in our integrand. We also have $\ln x$. Note that the derivative of $\ln x$ is $\frac{1}{x}$. So we have our term that needs to be substituted. Let,

$$u = \ln x$$

Taking differentials on both sides gives,

$$du = \frac{1}{x} dx$$

Let us now substitute these new expressions back into the original integral.

$$\int \frac{\tan(\ln x)}{x}\, dx = \int \tan(u)\, du \text{ (Since } \frac{1}{x} dx = du)$$

So now we have another problem. We have to integrate $\tan u$. There doesn't seem to be any term that we can really use for substitution in $\tan u$. Therefore, let's try to write $\tan u$ in terms of $\sin u$ and $\cos u$.

$$\int \tan u\, du = \int \frac{\sin u}{\cos u}\, du$$

Now we have $\cos u$ in the integrand and also its derivative $\sin u$. So, we have our term for substitution. Let,

$$w = \cos u$$

Taking differentials on both sides gives,

$$\begin{aligned} dw &= -\sin u\, du \\ -dw &= \sin u\, du \end{aligned}$$

Substituting these expressions back into the integral gives,

$$\begin{aligned}\int \tan u\, du = \int \frac{\sin u}{\cos u}\, du &= \int \frac{1}{w}\,(-dw)\\ &= -\int \frac{1}{w}\, dw\\ &= -\ln|w| + C\\ &= \ln\frac{1}{|w|} + C\end{aligned}$$

Now, we need to write the answer in terms of x. So we "reverse substitute" the values of w and u.

$$\ln\frac{1}{|w|} + C = \ln\frac{1}{|\cos u|} + C = \ln|\sec u| + C = \ln|\sec(\ln x)| + C.$$

Therefore,

$$\int \frac{\tan(\ln x)}{x}\, dx = \ln|\sec(\ln x)| + C.$$

□

Think about it: What would have happened if we had used $w = \sin u$ as our substitution in the previous problem? This is a natural question to ask since the derivative of $\sin u$ which is $\cos u$ is also present in the integrand.

3.2.3. Substitution in Definite Integrals

Here we look at definite integrals and see how the substitution method affect the way we solve the problem. The important thing we have to worry about when evaluating a definite integral using the method of substitution are its limits. In this regard we have two different ways to approach the problem.

3.2.3.1. **Method I - Changing the limits.** In this method we change the limits based on the substitution that we are using. Let us look at an example.

Example 3.9. Evaluate the definite integral.

$$\int_1^2 \frac{4x+12}{(x^2+6x+2)^2}\, dx$$

Solution. The derivative of x^2+6x+2 is $2x+6$ and note that $4x+12 = 2(2x+6)$. Hence, we have found our term that we need to substitute. Let,

$$u = x^2 + 6x + 2 \tag{3.10}$$

Taking differentials on both sides gives,

$$\begin{aligned} du &= (2x+6)\,dx \\ 2du &= (4x+12)\,dx \end{aligned}$$

Now, here comes the new part. The limits 1 and 2 are limits for the original variable of integration which was x. We have now introduced a new variable of integration viz. u, therefore, we need to find the limits for u. How do we find them? Look at equation 3.10, we will use that. Since,

$$u = x^2 + 6x + 2, \text{ therefore}$$
$$\text{When } x = 1, \quad u = 1^2 + 6 \cdot 1 + 2 = 9 \text{ and}$$
$$\text{When } x = 2, \quad u = 2^2 + 6 \cdot 2 + 2 = 18.$$

Thus the limits of u are 9 and 18. We now continue with our substitution,

$$\begin{aligned} \int_1^2 \frac{4x+12}{(x^2+6x+2)^2}\,dx &= \int_9^{18} \frac{2}{u^2}\,du \\ &= -\frac{2}{u}\bigg|_9^{18} \\ &= -\frac{2}{18} - (-\frac{2}{9}) \\ &= -\frac{1}{9} + \frac{2}{9} = \frac{1}{9} \end{aligned}$$

□

3.2.3.2. **Method II - Without changing the limits.** In this method we do not change the limits. Rather we evaluate as if it were an indefinite integral and then use the original limits in the answer. Let us apply this method to the previous problem.

Example 3.11. Evaluate the definite integral.

$$\int_1^2 \frac{4x+12}{(x^2+6x+2)^2}\,dx$$

Solution. We use the same substitution, but instead of finding the limits for u, we keep solving the indefinite integral

$$\begin{aligned} \int \frac{4x+12}{(x^2+6x+2)^2}\,dx &= \int \frac{2}{u^2}\,du, \qquad \text{where } u = x^2+6x+2 \\ &= -\frac{2}{u} + C \end{aligned}$$

Substituting the expression of u back into the answer gives,

$$\int \frac{4x+12}{(x^2+6x+2)^2}\,dx = -\frac{2}{x^2+6x+2} + C$$

Now, to evaluate the definite integral we use our original limits.

$$\begin{aligned}
\int_1^2 \frac{4x+12}{(x^2+6x+2)^2}\,dx &= -\frac{2}{x^2+6x+2}\Bigg|_1^2 \\
&= -\frac{2}{2^2+6\cdot 2+2} - \left(-\frac{2}{1^2+6\cdot 1+2}\right) \\
&= -\frac{2}{18} - \left(-\frac{2}{9}\right) \\
&= -\frac{1}{9} + \frac{2}{9} = \frac{1}{9}
\end{aligned}$$

It's not a coincidence that we end up with the same answer in both the methods.

□

Handout 9
MATH 252

Dibyajyoti Deb

9.1. Improper Integrals

9.1.1. Performance Criteria

(a) Evaluate an improper integral of the form $\displaystyle\int_a^\infty f(x)\,dx$.

9.1.2. Improper Integrals

The definite integrals that we have looked at so far have all had finite limits. The integrand was also continuous on this interval of integration. Now, what happens if either one or both limits are infinite or the integrand is not continuous on the interval of convergence? We call these types of integrals **improper integrals**. We deal first with improper integrals over infinite intervals where one or both endpoints may be infinite. Examples of such integrals are

$$\int_{-\infty}^{a} f(x)\,dx, \qquad \int_a^\infty f(x)\,dx, \qquad \int_{-\infty}^{\infty} f(x)\,dx$$

Now, how do we evaluate these integrals? The following result shows us a way.

Definition 9.1. The **improper integral** of $f(x)$ over $[a, \infty)$ is defined as the following limit (if it exists),

$$\int_a^\infty f(x)\,dx = \lim_{R\to\infty} \int_a^R f(x)\,dx$$

We say the above improper integral *converges* if the limit exists (and is finite) and that it *diverges* if the limit does not exist.

We can similarly define

$$\int_{-\infty}^{a} f(x)\,dx = \lim_{R\to-\infty}\int_{R}^{a} f(x)\,dx$$

and for a doubly infinite integral we can define it as a sum

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{\infty} f(x)\,dx$$

The above doubly infinite integral exists if both the integrals on the right converges.

Example 9.2. Determine whether the improper integral converges and if so, evaluate it.

$$\int_{4}^{\infty} e^{-3x}\,dx$$

Solution. We rewrite the integral as

$$\int_{4}^{\infty} e^{-3x}\,dx = \lim_{R\to\infty}\int_{4}^{R} e^{-3x}\,dx$$

Now we evaluate

$$\int e^{-3x}\,dx$$

as if it were a regular definite integral. Therefore,

$$\begin{aligned}\int_{4}^{R} e^{-3x}\,dx &= \frac{e^{-3x}}{-3}\Big|_{4}^{R}\\ &= -\frac{e^{-3R}}{3}+\frac{e^{-12}}{3}\end{aligned}$$

Now let us find the limit.

$$\begin{aligned}\lim_{R\to\infty}\int_{4}^{R} e^{-3x}\,dx &= \lim_{R\to\infty}\Big(-\frac{e^{-3R}}{3}+\frac{e^{-12}}{3}\Big)\\ &= \lim_{R\to\infty}\Big(-\frac{1}{3e^{3R}}+\frac{e^{-12}}{3}\Big)\end{aligned}$$

Now as $R\to\infty$, the expression $1/e^{3R}\to 0$. Hence,

$$\lim_{R\to\infty}\Big(-\frac{1}{3e^{3R}}+\frac{e^{-12}}{3}\Big) = \frac{e^{-12}}{3}$$

Therefore,

$$\int_{4}^{\infty} e^{-3x}\,dx = \frac{e^{-12}}{3}$$

and hence the integral converges.

$\square$

Let's look at another example.

Example 9.3. Determine whether the improper integral converges and if so, evaluate it.

$$\int_{-\infty}^{0} xe^{-x^2}\,dx$$

Solution. We rewrite the integral as

$$\int_{-\infty}^{0} xe^{-x^2}\,dx = \lim_{R\to-\infty}\int_{R}^{0} xe^{-x^2}\,dx$$

Now we evaluate

$$\int_{R}^{0} xe^{-x^2}\,dx$$

as if it were a regular definite integral. We use substitution in this case (Note that the derivative of x^2 is $2x$, and x is also present in the function). Using $u = x^2$ as our substitution we have,

$$\begin{aligned} du &= 2x\,dx \\ \frac{1}{2}\,du &= x\,dx \end{aligned}$$

The limits then change to R^2 and 0 (since $u = x^2$). The integral then becomes,

$$\begin{aligned} \frac{1}{2}\int_{R^2}^{0} e^{-u}\,du &= \frac{1}{2}(-e^{-u})\Big|_{R^2}^{0} \\ &= -\frac{1}{2}(1 - e^{-R^2}) \end{aligned}$$

Now, let us find the limit.

$$\lim_{R\to-\infty} -\frac{1}{2}(1 - e^{-R^2}) = -\frac{1}{2} \text{ (This is because } e^{-R^2} \to 0 \text{ as } R \to -\infty).$$

Thus the integral

$$\int_{-\infty}^{0} xe^{-x^2}\,dx \text{ converges.}$$

and

$$\int_{-\infty}^{0} xe^{-x^2}\,dx = -\frac{1}{2}$$

□

Now. let us look at a different type of improper integral. An integral over a finite interval $[a, b]$ is improper if the integrand becomes infinite at one or both of the endpoints of the interval. We evaluate these integrals the same as before, by evaluating a limit.

Definition 9.4. If $f(x)$ is continuous on $[a, b)$ but discontinuous at $x = b$, we define

$$\int_a^b f(x)\,dx = \lim_{R\to b-} \int_a^R f(x)\,dx$$

Similarly, if $f(x)$ is continuous on $(a, b]$ but discontinuous at $x = a$, then

$$\int_a^b f(x)\,dx = \lim_{R\to a+} \int_R^b f(x)\,dx$$

The above integrals converges if the limits exist and diverges otherwise.

Example 9.5. Evaluate the integral

$$\int_2^4 \frac{dx}{(x-2)^{1/3}}$$

Solution. The integrand here is discontinuous at the limit point 2. Thus,

$$\int_2^4 \frac{dx}{(x+2)^{1/3}} = \lim_{R\to 2+} \int_R^4 \frac{dx}{(x-2)^{1/3}}$$

Now we evaluate

$$\int_R^4 \frac{dx}{(x-2)^{1/3}}$$

as if it were a regular definite integral. We use substitution in this case. Let $u = x - 2$. Therefore, $du = dx$. The limits in this case become 2 and $R-2$ (as $u = x-2$). Thus the integral becomes,

$$\begin{aligned}\int_R^4 \frac{dx}{(x-2)^{1/3}} &= \int_{R-2}^2 \frac{du}{u^{1/3}} \\ &= \left.\frac{3u^{2/3}}{2}\right|_{R-2}^2 \\ &= \frac{3\cdot 2^{2/3}}{2} - \frac{3\cdot(R-2)^{2/3}}{2} \\ &= \frac{3}{2^{1/3}} - \frac{3\cdot(R-2)^{2/3}}{2}\end{aligned}$$

Now we apply the limit,

$$\lim_{R\to 2+} \int_R^4 \frac{dx}{(x-2)^{1/3}} = \lim_{R\to 2+} \left(\frac{3}{2^{1/3}} - \frac{3\cdot(R-2)^{2/3}}{2}\right) = \frac{3}{2^{1/3}}$$

□

9.2. Arc Length and Surface Area

9.2.1. Performance Criteria

(a) Set up an integral representing the length of a curve, given the formula.
(b) Set up an integral representing the area of a surface of revolution.

9.2.2. Arc Length

In this section we see how we can apply integrals to compute the length of a curve (which we call **arc length**). The goal is to approximate the length of the arc using small segments of straight lines. We skip the details here and look at the formula directly.

Theorem 9.6. *Assume that $f'(x)$ exists and is continuous on $[a,b]$. Then the arc length s of $y = f(x)$ over $[a,b]$ is equal to*

$$s = \int_a^b \sqrt{1+[f'(x)]^2}\,dx$$

Example 9.7. Find the arc length of $y = \frac{1}{12}x^3 + x^{-1}$ for $1 \le x \le 2$.

Solution. Here $f(x) = \frac{1}{12}x^3 + x^{-1}$. Therefore, $f'(x) = \frac{x^2}{4} - \frac{1}{x^2}$.
Hence,

$$\begin{aligned} 1 + [f'(x)]^2 &= 1 + \left(\frac{x^2}{4} - \frac{1}{x^2}\right)^2 \\ &= 1 + \frac{x^4}{16} - \frac{1}{2} + \frac{1}{x^4} \\ &= \frac{x^4}{16} + \frac{1}{2} + \frac{1}{x^4} \\ &= \left(\frac{x^2}{4} + \frac{1}{x^2}\right)^2 \end{aligned}$$

Thus,

$$\begin{aligned} s = \int_1^2 \sqrt{1 + [f'(x)]^2}\, dx &= \int_1^2 \left(\frac{x^2}{4} + \frac{1}{x^2}\right) dx \\ &= \frac{x^3}{12} - \frac{1}{x}\bigg|_1^2 \\ &= \left(\frac{8}{12} - \frac{1}{2}\right) - \left(\frac{1}{12} - 1\right) \\ &= \frac{13}{12} \end{aligned}$$

□

9.2.3. Area of a Surface of Revolution

We have seen earlier that if we rotate a curve around a vertical or horizontal line, we end up with a surface. In this section we see how we can apply integrals to compute the area of this surface of revolution.

Theorem 9.8. *Assume that $f(x) \geq 0$ and that $f'(x)$ exists and is continuous on $[a, b]$. The surface area S of the surface obtained by rotating the graph of $f(x)$ about the x-axis for $a \leq x \leq b$ is equal to*

$$S = 2\pi \int_a^b f(x)\sqrt{1 + f'(x)^2}\, dx$$

Example 9.9. Compute the surface area of revolution about the x-axis of the curve $y = x^3$ over the interval $[0, 2]$.

Solution. Here $f(x) = x^3$. Therefore, $f'(x) = 3x^2$. Hence,

$$\sqrt{1 + f'(x)^2} = \sqrt{1 + 9x^4}$$

Therefore,

$$S = 2\pi \int_0^2 f(x)\sqrt{1 + f'(x)^2}\,dx = 2\pi \int_0^2 x^3\sqrt{1 + 9x^4}\,dx$$

We use the substitution $u = 1 + 9x^4$ here. Therefore,

$$\begin{aligned} du &= 36x^3\,dx \\ \frac{1}{36}\,du &= x^3\,dx \end{aligned}$$

The limits in this case become

$$\text{When } x = 0, u = 1 + 9 \cdot 0^4 = 1.$$
$$\text{When } x = 2, u = 1 + 9 \cdot 2^4 = 145.$$

The integral then becomes,

$$\begin{aligned} 2\pi \int_0^2 x^3\sqrt{1 + 9x^4}\,dx &= \frac{2\pi}{36}\int_1^{145} \sqrt{u}\,du \\ &= \frac{\pi}{9} \cdot \left.\frac{u^{3/2}}{3}\right|_1^{145} \\ &= \frac{\pi}{27}(145^{3/2} - 1) \end{aligned}$$

$\square$

Handout 7
MATH 252

Dibyajyoti Deb

7.1. Techniques of Integration: Integration by Parts

7.1.1. Performance Criteria

(a) Compute an anti-derivative using integration by parts.

7.1.2. Integration by Parts

As I have mentioned before, we need to use different methods to integrate different types of functions, unlike finding their derivatives which relies on very few techniques (like Product, Quotient or Chain Rule).

In this section we look at probably the most important technique to integrate certain types of functions. It is called **Integration by Parts** and it is analogous to the Product Rule for derivatives that you seen earlier. It is in fact derived from the Product Rule.

If u and v are functions of x, then by the Product Rule,

$$(uv)' = uv' + u'v$$

Integrating both sides with respect to x we have,

$$\begin{aligned} uv &= \int uv'\,dx + \int u'v\,dx \\ \int uv'\,dx &= uv - \int u'v\,dx \end{aligned}$$

Thus,

Integration by Parts Formula : If u and v are functions of x, then,

$$\int uv'\,dx = uv - \int u'v\,dx \tag{7.1}$$

Since we are integrating a product uv', hence the reason why integration by parts is analogous to the product rule.

The important step in this method is making the right choice for u and v' such that finding u' (which we find by differentiating u) and v (which we find by integrating v') and integrating $u'v$ is easy enough.

Example 7.2. Evaluate

$$\int xe^x\,dx$$

Solution. We have to choose u and v', such that finding u' and v and integrating $u'v$ is easy enough as I mentioned before. In this regard having $u = x$ and $v' = e^x$ is the right choice as $u' = 1$, $v = e^x$ and integrating $u'v = e^x$ is easy. Thus,

$$u = x \;\Rightarrow\; u' = 1 \text{ (by differentiating } u)$$

$$v' = e^x \;\Rightarrow\; v = e^x \text{ (by integrating } v')$$

Thus by Equation 7.1,

$$\begin{aligned} \int xe^x &= xe^x - \int e^x\,dx \\ &= xe^x - e^x + C \end{aligned}$$

□

In the previous example, what would have happened if we had chosen $u = e^x$ and $v' = x$? By differentiating u and integrating v' we have, $u' = e^x$ and $v = x^2/2$. However, it is not so easy to integrate $uv' = \dfrac{e^x x^2}{2}$. Thus, $u = e^x$ and $v' = x$ turn out to be bad choices for u and v.

Example 7.3. Evaluate

$$\int \ln x\,dx$$

Solution. The function $\ln x$ doesn't seem to be a product of two functions, however we can think of $\ln x$ as $\ln x \cdot 1$. Hence we can choose $u = \ln x$ and $v = 1$. Then,

$$u = \ln x \;\Rightarrow\; u' = \frac{1}{x} \text{ (by differentiating } u\text{)}$$

$$v' = 1 \;\Rightarrow\; v = x \text{ (by integrating } v'\text{)}$$

Thus by Equation 7.1,

$$\begin{aligned}\int \ln x\,dx &= x\ln x - \int \frac{1}{x}\cdot x\,dx \\ &= x\ln x - \int dx \\ &= x\ln x - x + C\end{aligned}$$

$\square$

Example 7.4. Evaluate

$$\int x^2 e^x\,dx$$

Solution. Here we choose $u = x^2$ and $v' = e^x$, then

$$u = x^2 \;\Rightarrow\; u' = 2x \text{ (by differentiating } u\text{)}$$

$$v' = e^x \;\Rightarrow\; v = e^x \text{ (by integrating } v'\text{)}$$

This is the right choice for u and v' because when we find u' by differentiating then the degree of x keeps decreasing as we apply the power rule to x^2. By Equation 7.1 we have,

$$\begin{aligned}\int x^2e^x &= x^2e^x - \int 2xe^x\,dx \\ &= x^2e^x - 2\int xe^x\,dx\end{aligned}$$

To evaluate $\int xe^x\,dx$, we would have to use untegration by parts again with $u = x$ and $v' = e^x$. However, since we have already evaluated this integral in Example 7.2, therefore, we can use that result. Hence,

$$\begin{aligned}\int x^2e^x\,dx &= x^2e^x - 2(xe^x - e^x) + C \\ &= x^2e^x - 2xe^x + e^x + C\end{aligned}$$

$\square$

Example 7.5. Evaluate

$$\int e^x \sin x \, dx$$

Solution. We choose $u = e^x$ and $v' = \sin x$. Then,

$$u = e^x \;\Rightarrow\; u' = e^x \text{ (by differentiating } u\text{)}$$

$$v' = \sin x \;\Rightarrow\; v = -\cos x \text{ (by integrating } v'\text{)}$$

By Equation 7.1 we have,

$$\begin{aligned}\int e^x \sin x \, dx &= -e^x \cos x - \int e^x(-\cos x)\, dx \\ &= -e^x \cos x + \int e^x \cos x \, dx\end{aligned}$$

Now it seems like we have to integrate $e^x \cos x$. For this we choose $u = e^x$ and $v' = \cos x$. Then,

$$u = e^x \;\Rightarrow\; u' = e^x \text{ (by differentiating } u\text{)}$$

$$v' = \cos x \;\Rightarrow\; v = \sin x \text{ (by integrating } v'\text{)}$$

By Equation 7.1 we have,

$$\int e^x \sin x \, dx = -e^x \cos x + e^x \sin x - \int e^x \sin x \, dx$$

It seems like we are moving in a circle now since we end with the integration of $e^x \sin x$ on the right side too. However if we let $I = \int e^x \sin x \, dx$, then

$$\begin{aligned} I &= -e^x \cos x + e^x \sin x - I \\ 2I &= e^x(\sin x - \cos x) \\ I &= \frac{e^x(\sin x - \cos x)}{2}\end{aligned}$$

Thus,

$$\int e^x \sin x \, dx = \frac{e^x(\sin x - \cos x)}{2} + C$$

□

Now, what happens when we have a definite integral? Does integration by parts affect the way we evaluate a definite integral?

Integration by Parts for definite integrals

(7.6) $$\int_a^b uv'\,dx = uv\Big|_a^b - \int_a^b u'v\,dx$$

Example 7.7. Evaluate
$$\int_0^3 xe^{4x}\,dx$$

Solution. We choose $u = x$ and $v' = e^{4x}$. Then,
$$u = x \;\Rightarrow\; u' = 1 \text{ (by differentiating } u)$$
$$v' = e^{4x} \;\Rightarrow\; v = \frac{e^{4x}}{4} \text{ (by integrating } v')$$
Therefore, by Equation 7.6 we have,
$$\int_0^3 xe^{4x}\,dx \;=\; x \cdot \frac{e^{4x}}{4}\Big|_0^3 - \int_0^3 \frac{e^{4x}}{4}\,dx$$

Using the substitution $u = 4x$,
$$\int \frac{e^{4x}}{4}\,dx = \frac{e^{4x}}{16} + C$$
Hence,
$$\begin{aligned}\int_0^3 xe^{4x}\,dx &= \left(3 \cdot \frac{e^{12}}{4} - 0 \cdot \frac{e^0}{4}\right) - \frac{e^{4x}}{16}\Big|_0^3 \\ &= \frac{3e^{12}}{4} - \left(\frac{e^{12}}{16} - \frac{e^0}{16}\right) \\ &= \frac{3e^{12}}{4} - \frac{e^{12}}{16} + \frac{1}{16} = \frac{11e^{12}}{16} + \frac{1}{16}\end{aligned}$$

□

7.2. Trigonometric Integrals

7.2.1. Performance Criteria

(a) Compute the anti-derivative of functions involving powers of trigonometric functions.

7.2.2. Trigonometric Integrals

In this section we look at methods by which we can integrate functions which are powers of trigonometric functions. We will be looking at powers of sine, cosine, tangent and secant. Powers of cotangent and cosecant can integrated using the same methods for tangent and secant.

7.2.2.1. **Powers of Sine and Cosine.**

(1) **Powers of Sine** - We look at methods for evaluating

$$\int \sin^n x\,dx$$

for a positive integer n.

To evaluate integrals of this form we use the reduction formula for Sine.

Reduction formula for Sine: For a positive integer n,

$$\int \sin^n x\,dx = -\frac{1}{n}\sin^{n-1}x\cos x + \frac{n-1}{n}\int \sin^{n-2}x\,dx$$

If we denote $\int \sin^n x\,dx$ by I_n, then the above formula can be rewritten as

$$I_n = -\frac{1}{n}\sin^{n-1}x\cos x + \frac{n-1}{n}I_{n-2}$$

Example 7.8. Evaluate

$$\int \sin^5 x\,dx$$

Solution. Here we have to find I_5. Using the reduction formula for Sine we have,

$$I_5 = -\frac{1}{5}\sin^4 x\cos x + \frac{4}{5}I_3 \tag{7.9}$$

which means we have to find I_3 now. By the same reduction formula,

$$I_3 = -\frac{1}{3}\sin^2 x\cos x + \frac{2}{3}I_1 \tag{7.10}$$

So we have to find I_1 now. I_1 is the "base case" and it can be computed by

$$I_1 = \int \sin x\,dx = -\cos x + C \tag{7.11}$$

Using this in 7.10 we have,

$$I_3 = -\frac{1}{3}\sin^2 x\cos x + \frac{2}{3}(-\cos x) + C$$

Using this in 7.9 we have,

$$\begin{aligned} I_5 &= -\frac{1}{5}\sin^4 x\cos x + \frac{4}{5}(-\frac{1}{3}\sin^2 x\cos x + \frac{2}{3}(-\cos x)) + C \\ &= -\frac{1}{5}\sin^4 x\cos x - \frac{4}{15}\sin^2 x\cos x - \frac{8}{15}\cos x + C \end{aligned}$$

$\square$

(2) **Powers of Cosine** - We look at methods for evaluating

$$\int \cos^n x\,dx$$

for a positive integer n.

To evaluate integrals of this form we use the reduction formula for Cosine.

Reduction formula for Cosine: For a positive integer n,

$$\int \cos^n x\,dx = \frac{1}{n}\cos^{n-1} x\sin x + \frac{n-1}{n}\int \cos^{n-2} x\,dx$$

If we denote $\int \cos^n x\,dx$ by I_n, then the above formula can be rewritten as

$$I_n = \frac{1}{n}\cos^{n-1} x\sin x + \frac{n-1}{n}I_{n-2}$$

Example 7.12. Evaluate

$$\int \cos^6 x\,dx$$

Solution. Here we have to find I_6. Using the reduction formula for Sine we have,

$$I_6 = \frac{1}{6}\cos^5 x \sin x + \frac{5}{6} I_4 \tag{7.13}$$

which means we have to find I_4 now. By the same reduction formula,

$$I_4 = \frac{1}{4}\cos^3 x \sin x + \frac{3}{4} I_2 \tag{7.14}$$

So we have to find I_2 now. I_2 is the "base case" and it can be computed by

$$I_2 = \int \cos^2 x \, dx \tag{7.15}$$

Now,

$$\cos^2 x = \frac{1 + \cos 2x}{2}$$

by the double angle formula for cosine. Hence,

$$\begin{aligned}
\int \cos^2 x \, dx &= \int \frac{1 + \cos 2x}{2} \, dx \\
&= \int (\frac{1}{2} + \frac{\cos 2x}{2}) \, dx \\
&= \frac{x}{2} + \frac{\sin 2x}{4} + C
\end{aligned}$$

Using this in 7.14 we have,

$$\begin{aligned}
I_4 &= \frac{1}{4}\cos^3 x \sin x + \frac{3}{4}(\frac{x}{2} + \frac{\sin 2x}{4}) + C \\
&= \frac{1}{4}\cos^3 x \sin x + \frac{3x}{8} + \frac{3\sin 2x}{16} + C
\end{aligned}$$

Using this in 7.13 we have,

$$\begin{aligned}
I_6 &= \frac{1}{6}\cos^5 x \sin x + \frac{5}{6}(\frac{1}{4}\cos^3 x \sin x + \frac{3x}{8} + \frac{3\sin 2x}{16}) + C \\
&= \frac{1}{6}\cos^5 x \sin x + \frac{5}{24}\cos^3 x \sin x + \frac{5x}{16} + \frac{5\sin 2x}{32} + C
\end{aligned}$$

□

As is evidenced from the previous two examples, the base case turns out to be either I_1 or I_2 for both the trigonometric functions. If it's I_2, then we have to integrate either $\sin^2 x$ or

$\cos^2 x$. We have already seen how to integrate $\cos^2 x$. $\sin^2 x$ can be integrated the same way.

$$I_2 = \int \sin^2 x\,dx$$

Now,

$$\sin^2 x = \frac{1-\cos 2x}{2}$$

by the double angle formula for sine. Hence,

$$\begin{aligned}\int \sin^2 x\,dx &= \int \frac{1-\cos 2x}{2}\,dx \\ &= \int (\frac{1}{2} - \frac{\cos 2x}{2})\,dx \\ &= \frac{x}{2} - \frac{\sin 2x}{4} + C\end{aligned}$$

7.2.2.2. **Products of powers of sine and cosine.** Here we look at methods for evaluating

$$\int \sin^m x \cos^n x\,dx$$

for positive integers m and n.

(1) **At least one of m, n are odd** - Let us say m is odd (the same method could be applied if n is odd or if both m and n are odd). In this case we write the integral as

$$\int \sin^m x \cos^n x\,dx = \int \sin x \sin^{m-1} x \cos^n x\,dx$$

Now since m is odd, therefore $m-1$ is even and hence

$$\begin{aligned}\int \sin x \sin^{m-1} x \cos^n x\,dx &= \int \sin x\,(\sin^2 x)^{\frac{m-1}{2}} x \cos^n x\,dx \\ &= \int \sin x(1-\cos^2 x)^{\frac{m-1}{2}} x \cos^n x\,dx\end{aligned}$$

Now we use the substitution $u = \cos x$ and then $du = -\sin x\,dx$, and therefore,

$$\int \sin x(1-\cos^2 x)^{\frac{m-1}{2}} x \cos^n x\,dx = -\int (1-u^2)^{\frac{m-1}{2}} u^n\,du$$

Now the above integral can be technically evaluated by expanding $(1-u^2)^{\frac{m-1}{2}}$ and then multiplying with u^n and then using the power rule for each term. Let us look at an example of this.

Example 7.16. Evaluate

$$\int \sin^5 x \cos^6 x \, dx$$

Solution. Here we rewrite the integral as

$$\int \sin^5 x \cos^6 x \, dx = \int \sin x \sin^4 x \cos^6 x \, dx$$

Now since $\sin^2 x = 1 - \cos^2 x$, therefore,

$$\int \sin x \sin^4 x \cos^6 x \, dx = \int \sin x (1 - \cos^2 x)^2 \cos^6 x \, dx$$

Using the substitution $u = \cos x$, we have $du = -\sin x \, dx$. Therefore,

$$\begin{aligned}
\int \sin x (1 - \cos^2 x)^2 \cos^6 x \, dx &= -\int (1 - u^2)^2 u^6 \, du \\
&= -\int (1 - 2u^2 + u^4) u^6 \, du \\
&= -\int (u^6 - 2u^8 + u^{10}) \, du \\
&= -\frac{u^7}{7} + \frac{2u^9}{9} - \frac{u^{11}}{11} + C
\end{aligned}$$

Substituting u back gives,

$$\int \sin^5 x \cos^6 x \, dx = -\frac{\cos^7 x}{7} + \frac{2\cos^9 x}{9} - \frac{\cos^{11} x}{11} + C$$

□

(2) **Both m, n are even** - Let $m \leq n$ (the same method applies when $n \leq m$). In this case we write the integral as,

$$\int \sin^m x \cos^n x \, dx = \int (\sin^2 x)^{\frac{m}{2}} \cos^n x \, dx$$

Now since $\sin^2 x = 1 - \cos^2 x$, therefore,

$$\int (\sin^2 x)^{\frac{m}{2}} \cos^n x \, dx = \int (1 - \cos^2 x)^{\frac{m}{2}} \cos^n x \, dx$$

Now the above integral can be evaluated by expanding $(1 - \cos^2 x)^{\frac{m}{2}}$ and then multiplying with $\cos^n x$ and then using the reduction formula for cosine for each term. Let us look at an example of this.

Example 7.17. Evaluate

$$\int \sin^4 x \cos^6 x \, dx$$

Solution. We rewrite the integral as

$$\begin{aligned}
\int \sin^4 x \cos^6 x \, dx &= \int (\sin^2 x)^2 \cos^6 x \, dx \\
&= \int (1 - \cos^2 x)^2 \cos^6 x \, dx \\
&= \int (1 - 2\cos^2 x + \cos^4 x) \cos^6 x \, dx \\
&= \int (\cos^6 x - 2\cos^8 x + \cos^{10} x) \, dx \\
&= \int \cos^6 x \, dx - 2\int \cos^8 x \, dx + \int \cos^{10} x \, dx
\end{aligned}$$

Each of these integrals can be evaluated by using the reduction formula for cosine. First, let us compute

$$I_{10} = \int \cos^{10} \, dx$$

We use the reduction formula for cosine,

$$I_{10} = \frac{1}{10}\cos^9 x \sin x + \frac{9}{10} I_8 \tag{7.18}$$

which means we have to find I_8 now. By the same reduction formula,

$$I_8 = \frac{1}{8}\cos^7 x \sin x + \frac{7}{8} I_6 \tag{7.19}$$

which means we have to find I_6 now which we have already calculated earlier in 7.16. Therefore, we have

$$\begin{aligned}
I_8 &= \frac{1}{8}\cos^7 x \sin x + \frac{7}{8}\left(\frac{1}{6}\cos^5 x \sin x + \frac{5}{24}\cos^3 x \sin x + \frac{5x}{16} + \frac{5\sin 2x}{32}\right) + C \\
&= \frac{1}{8}\cos^7 x \sin x + \frac{7}{48}\cos^5 x \sin x + \frac{35}{192}\cos^3 x \sin x + \frac{35x}{128} + \frac{35\sin 2x}{256} + C
\end{aligned}$$

Using this in Equation 7.18 we have,

$$\begin{aligned}
I_{10} &= \frac{1}{10}\cos^9 x \sin x + \frac{9}{10}\left(\frac{1}{8}\cos^7 x \sin x + \frac{7}{48}\cos^5 x \sin x + \frac{35}{192}\cos^3 x \sin x + \right. \\
&\qquad \left. \frac{35x}{128} + \frac{35\sin 2x}{256}\right) + C \\
&= \frac{1}{10}\cos^9 x \sin x + \frac{9}{80}\cos^7 x \sin x + \frac{63}{480}\cos^5 x \sin x + \frac{315}{1920}\cos^3 x \sin x +
\end{aligned}$$

$$\frac{315x}{1280} + \frac{315\sin 2x}{2560} + C$$

Thus,

$$\int \sin^4 x \cos^6 x \, dx = I_6 - 2I_8 + I_{10} + C$$

where,

$$I_6 = \frac{1}{6}\cos^5 x \sin x + \frac{5}{24}\cos^3 x \sin x + \frac{5x}{16} + \frac{5\sin 2x}{32}$$

$$I_8 = \frac{1}{8}\cos^7 x \sin x + \frac{7}{48}\cos^5 x \sin x + \frac{35}{192}\cos^3 x \sin x + \frac{35x}{128} + \frac{35\sin 2x}{256}$$

and

$$I_{10} = \frac{1}{10}\cos^9 x \sin x + \frac{9}{80}\cos^7 x \sin x + \frac{63}{480}\cos^5 x \sin x + \frac{315}{1920}\cos^3 x \sin x + \frac{315x}{1280} + \frac{315\sin 2x}{2560}$$

□

7.2.2.3. **Powers of Tangent and Secant.**

(1) **Powers of Tangent** - We look at methods for evaluating

$$\int \tan^n x \, dx$$

for a positive integer n.

To evaluate integrals of this form we use the reduction formula for Tangent.

Reduction formula for Tangent: For a positive integer n,

$$\int \tan^n x \, dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x \, dx$$

If we denote $\int \tan^n x \, dx$ by I_n, then the above formula can be rewritten as

$$I_n = \frac{\tan^{n-1} x}{n-1} - I_{n-2}$$

Example 7.20. Evaluate

$$\int \tan^5 x \, dx$$

Solution. Here we have to find I_5. Using the reduction formula for Tangent we have,

$$I_5 = \frac{\tan^4 x}{4} - I_3 \tag{7.21}$$

which means we have to find I_3 now. By the same reduction formula,

$$I_3 = \frac{\tan^2 x}{2} - I_1 \tag{7.22}$$

So we have to find I_1 now. I_1 is the "base case" and it can be computed by

$$I_1 = \int \tan x\, dx = \int \frac{\sin x}{\cos x}\, dx$$

If we use the substitution $u = \cos x$, then $du = -\sin x\, dx$. Hence,

$$\begin{aligned}\int \tan x\, dx &= -\int \frac{1}{u}\, du \\ &= -\ln|u| + C \\ &= -\ln|\cos x| + C = \ln|\sec x| + C\end{aligned}$$

Using this in 7.22 we have,

$$I_3 = \frac{\tan^2 x}{2} - \ln|\sec x| + C$$

Using this in 7.21 we have,

$$\begin{aligned}I_5 &= \frac{\tan^4 x}{4} - (\frac{\tan^2 x}{2} - \ln|\sec x|) + C \\ &= \frac{\tan^4 x}{4} - \frac{\tan^2 x}{2} + \ln|\sec x| + C\end{aligned}$$

□

If the base case is I_2, then we have to evaluate

$$\begin{aligned}I_2 = \int \tan^2 x\, dx &= \int (\sec^2 x - 1)\, dx \\ &= \int \sec^2 x\, dx - \int dx \\ &= \tan x - x + C\end{aligned}$$

(2) **Powers of Secant** - We look at methods for evaluating

$$\int \sec^n x\, dx$$

for a positive integer n.

To evaluate integrals of this form we use the reduction formula for Secant.

Reduction formula for Secant: For a positive integer n,

$$\int \sec^n x\,dx = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2} x\,dx$$

If we denote $\int \tan^n x\,dx$ by I_n, then the above formula can be rewritten as

$$I_n = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1} I_{n-2}$$

Example 7.23. Evaluate

$$\int \sec^6 x\,dx$$

Solution. Here we have to find I_6. Using the reduction formula for Secant we have,

$$I_6 = \frac{\sec^4 x \tan x}{5} + \frac{4}{5} I_4 \tag{7.24}$$

which means we have to find I_4 now. By the same reduction formula,

$$I_4 = \frac{\sec^2 x \tan x}{3} + \frac{2}{3} I_2 \tag{7.25}$$

So we have to find I_2 now. I_2 is the "base case" and it can be computed by

$$I_2 = \int \sec^2 x\,dx = \tan x + C \tag{7.26}$$

Using this in 7.25 we have,

$$I_4 = \frac{\sec^2 x \tan x}{3} + \frac{2}{3}\tan x + C$$

Using this in 7.24 we have,

$$\begin{aligned} I_6 &= \frac{\sec^4 x \tan x}{5} + \frac{4}{5}\left(\frac{\sec^2 x \tan x}{3} + \frac{2}{3}\tan x\right) + C \\ &= \frac{\sec^4 x \tan x}{5} + \frac{5 \sec^2 x \tan x}{15} + \frac{8}{15}\tan x + C \end{aligned}$$

□

If the base case is I_1, then we have to evaluate

$$I_1 = \int \sec x\, dx$$

We multiply both the numerator and the denominator by $\sec x + \tan x$. We then have,

$$\begin{aligned}\int \sec x\, dx &= \int \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x}\, dx \\ &= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\, dx\end{aligned}$$

We use the substitution $u = \sec x + \tan x$, then $du = \sec x \tan x + \sec^2 x\, dx$. Therefore,

$$\begin{aligned}\int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\, dx &= \frac{1}{u}\, du \\ &= \ln|u| + C\end{aligned}$$

Substituting u back we have,

$$\int \sec x\, dx = \ln|\sec x + \tan x| + C$$

7.2.2.4. **Products of powers of tangent and secant.** Here we look at methods for evaluating

$$\int \tan^m x \sec^n x\, dx$$

for positive integers m and n.

(1) m **is odd,** n **anything** - In this case we write the integral as

$$\int \tan^m x \sec^n x\, dx = \int \tan^{m-1} x \sec^{n-1} x(\sec x \tan x)\, dx$$

Since m is odd, therefore, $m - 1$ is even and hence we can use the identity $\tan^2 x = sec^2x - 1$ to have,

$$\begin{aligned}\int \tan^m x \sec^n x\, dx &= \int (\tan^2 x)^{\frac{m-1}{2}} \sec^{n-1} x(\sec x \tan x)\, dx \\ &= \int (\sec^2 x - 1)^{\frac{m-1}{2}} \sec^{n-1} x(\sec x \tan x)\, dx\end{aligned}$$

We use the substitution $u = \sec x$ and so $du = \sec x \tan x \, dx$. Therefore,

$$\int \tan^m x \sec^n x \, dx = \int (u^2 - 1)^{\frac{m-1}{2}} u^{n-1} \, du$$

Now the above integral can be technically evaluated by expanding $(u^2 - 1)^{\frac{m-1}{2}}$ and then multiplying with u^n and then using the power rule for each term. Let us look at an example of this.

Example 7.27. Evaluate

$$\int \tan^5 x \sec^4 x \, dx$$

Solution. We rewrite the integral as

$$\int \tan^5 x \sec^4 x \, dx = \int \tan^4 x \sec^3 x (\sec x \tan x) \, dx$$

We now use the substitution $u = \sec x$ and so $du = \sec x \tan x \, dx$. We also use the identity $\tan^2 x = \sec^2 x - 1$. Therefore,

$$\begin{aligned}
\int \tan^5 x \sec^4 x \, dx &= \int \tan^4 x \sec^3 x (\sec x \tan x) \, dx \\
&= \int (\sec^2 x - 1)^2 \sec^3 x (\sec x \tan x) \, dx \\
&= \int (u^2 - 1)^2 u^3 \, du \\
&= \int (u^4 - 2u^2 + 1) u^3 \, du \\
&= \int (u^7 - 2u^5 + u^3) \, du \\
&= \frac{u^8}{8} - \frac{u^6}{3} + \frac{u^4}{4} + C
\end{aligned}$$

Substituting u back gives,

$$\int \tan^5 x \sec^4 x \, dx = \frac{\sec^8 x}{8} - \frac{\sec^6 x}{3} + \frac{\sec^4 x}{4} + C$$

□

(2) m **is even,** n **anything** - In this case we rewrite the integral as

$$\int \tan^m x \sec^n x \, dx = \int (\tan^2 x)^{\frac{m}{2}} \sec^n x \, dx$$

Using the identity $\tan^2 x = \sec^2 x - 1$ we have,

$$\int \tan^m x \sec^n x\, dx = \int (\sec^2 x - 1)^{\frac{m}{2}} \sec^n x\, dx$$

Now the above integral can be evaluated by expanding $(\sec^2 x - 1)^{\frac{m}{2}}$ and then multiplying with $\sec^n x$ and then using the reduction formula for secant for each term. Let us look at an example of this.

Example 7.28. Evaluate

$$\int \tan^4 x \sec^2 x\, dx$$

Solution. We rewrite the integral as

$$\begin{aligned}
\int \tan^4 x \sec^2 x\, dx &= \int (\tan^2 x)^2 \sec^3\ dx \\
&= \int (\sec^2 x - 1)^2 \sec^2\ dx \\
&= \int (\sec^4 x - 2\sec^2 x + 1) \sec^2\ dx \\
&= \int (\sec^6 x - 2\sec^4 x + \sec^2 x)\, dx \\
&= \int \sec^6 x\, dx - 2\int \sec^4 x\, dx + \int \sec^2 x\, dx
\end{aligned}$$

Each of these integrals can be evaluated by using the reduction formula for secant. First, let us compute

$$I_6 = \int \sec^6\ dx$$

We have already computed this in Example 7.23.

$$\int \sec^6 x\, dx = \frac{\sec^4 x \tan x}{5} + \frac{5\sec^2 x \tan x}{15} + \frac{8}{15}\tan x + C$$

We have also computed $I_4 = \int \sec^4 x\, dx$ and $\int \sec^2 x\, dx$ in Example 7.23.

$$\int \sec^4 x\, dx = \frac{\sec^2 x \tan x}{3} + \frac{2}{3}\tan x + C$$

and

$$\int \sec^2 x\, dx = \tan x + C$$

Therefore,

$$\int \tan^4 x \sec^2 x \, dx = I_6 - 2I_4 + I_2 + C$$

where,

$$I_6 = \frac{\sec^4 x \tan x}{5} + \frac{5 \sec^2 x \tan x}{15} + \frac{8}{15} \tan x$$

$$I_4 = \frac{\sec^2 x \tan x}{3} + \frac{2}{3} \tan x$$

$$I_2 = \tan x$$

□

Handout 8
MATH 252

Dibyajyoti Deb

8.1. Trigonometric Substitution

8.1.1. Performance Criteria

(a) Evaluate an integral containing one of the forms $a^2 - x^2, a^2 + x^2, x^2 - a^2$ or the square root of any of those forms.

8.1.2. Trigonometric Substitution

In this section we look at methods by which we can evaluate integrals containing certain forms. We will use substitution but involving trigonometric functions. We will also use the following well known trigonometric identities. For any angle θ,

$$\begin{aligned} \sin^2\theta + \cos^2\theta &= 1 \\ 1 + \tan^2\theta &= \sec^2\theta \\ \sec^2\theta - 1 &= \tan^2\theta \end{aligned}$$

Form 1. Function involving $a^2 - x^2$. When we have a function that involves this form then we use the substitution

$$x = a\sin\theta$$

and we continue with the integration. We look at an example illustrating this substitution.

Example 8.1. Evaluate

$$\int \sqrt{9 - x^2}\,dx$$

Solution. We see that we have the form $a^2 - x^2$ in the expression, with $a = 3$. Hence, we use the substitution,

$$x = 3\sin\theta$$

Therefore,

$$dx = 3\cos\theta\, d\theta$$

Substituting back we have,

$$\begin{aligned}\int \sqrt{9 - 9\sin^2\theta}\; 3\cos\theta\, d\theta &= \int \sqrt{9\cos^2\theta}\; 3\cos\theta\, d\theta \\ &= 9\int \cos^2\theta\; d\theta\end{aligned}$$

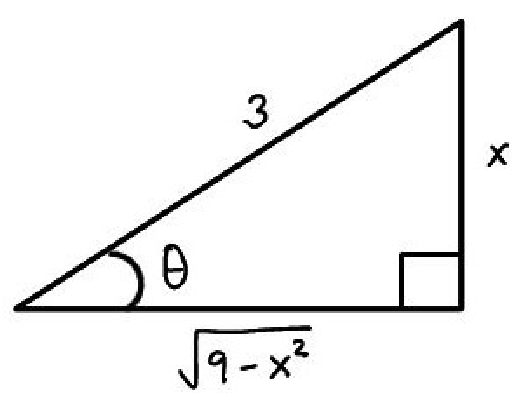
We now use the double angle formula for $\cos^2\theta$.

$$\cos^2\theta = \frac{1+\cos 2\theta}{2}$$

The integral then becomes,

$$\begin{aligned}9\int \cos^2\theta\, d\theta &= \frac{9}{2}\int (1+\cos 2\theta)\, d\theta \\ &= \frac{9}{2}(\int d\theta + \int \cos 2\theta\, d\theta) \\ &= \frac{9}{2}\left(\theta + \frac{\sin 2\theta}{2}\right) + C \\ &= \frac{9\theta}{2} + \frac{9\sin 2\theta}{4} + C\end{aligned}$$

Now we have to write this answer in terms of x. Since $x = 3\sin\theta$, therefore, $\theta = \sin^{-1}(x/3)$. To find $\sin 2\theta$ in terms of x, we have to complete the right triangle as below. Since $x/3 = \sin\theta$, therefore the opposite is x and the hypotenuse is 3. The adjacent is therefore $\sqrt{9-x^2}$.



From the picture above,

$$\cos\theta = \frac{\sqrt{9-x^2}}{3}$$

Now

$$\sin 2\theta = 2\sin\theta\cos\theta = 2\cdot\frac{x}{3}\cdot\frac{\sqrt{9-x^2}}{3} = \frac{2x\sqrt{9-x^2}}{9}$$

Therefore,

$$\begin{aligned}\int\sqrt{9-x^2}\,dx &= \frac{9\theta}{2}+\frac{9\sin 2\theta}{4}+C\\ &= \frac{9}{2}\sin^{-1}(\frac{x}{3})+\frac{9}{4}\cdot\frac{2x\sqrt{9-x^2}}{9}+C\\ &= \frac{9}{2}\sin^{-1}(\frac{x}{3})+\frac{x\sqrt{9-x^2}}{2}+C\end{aligned}$$

$\square$

Form 2. Function involving a^2+x^2. When we have a function that involves this form then we use the substitution

$$x = a\tan\theta$$

and we continue with the integration. We look at an example illustrating this substitution.

Example 8.2. Evaluate

$$\int_0^1 \frac{dx}{(4+9x^2)^2}$$

Solution. This is a definite integral. Let us first try to integrate and not worry about the limits until the end. Here it seems like the expression has the form a^2+x^2 in it, however it's not so obvious since we have a 9 multiplied with x^2. We first need to get rid of this 9. For this we factor out the 9 from the expression.

$$\begin{aligned}(4+9x^2)^2 &= \left(9\left(\frac{4}{9}+x^2\right)\right)^2\\ &= 9^2\left(\frac{4}{9}+x^2\right)^2\end{aligned}$$

Therefore,

$$\int\frac{dx}{(4+9x^2)^2} = \int\frac{dx}{9^2\left(\frac{4}{9}+x^2\right)^2}$$

Now, this resembles the form a^2+x^2 with $a=2/3$. We use the substitution

$$x = \frac{2}{3}\tan\theta$$

Hence

$$dx = \frac{2}{3}\sec^2\theta\, d\theta$$

Substituting back we have,

$$\begin{aligned}
\int \frac{dx}{9^2\left(\frac{4}{9}+x^2\right)^2} &= \frac{1}{81}\int \frac{\frac{2}{3}\tan^2\theta\, d\theta}{(\frac{4}{9}+\frac{4}{9}\tan^2\theta)^2} \\
&= \frac{2}{243}\int \frac{\sec^2\ d\theta}{\frac{16}{81}(1+\tan^2\theta)^2} \\
&= \frac{1}{24}\int \frac{\sec^2\theta\, d\theta}{\sec^4\theta} \\
&= \frac{1}{24}\int \frac{d\theta}{\sec^2\theta} \\
&= \frac{1}{24}\int \cos^2\ d\theta
\end{aligned}$$

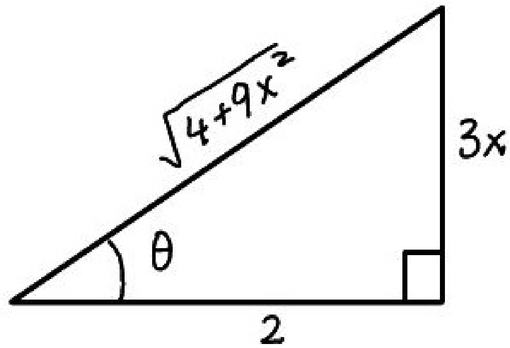
We now use the double angle formula for $\cos^2\theta$.

$$\cos^2\theta = \frac{1+\cos 2\theta}{2}$$

The integral then becomes,

$$\begin{aligned}
\frac{1}{24}\int \cos^2\theta\, d\theta &= \frac{1}{48}\int(1+\cos 2\theta)\, d\theta \\
&= \frac{1}{48}(\int d\theta + \int \cos 2\theta\, d\theta) \\
&= \frac{1}{48}\left(\theta + \frac{\sin 2\theta}{2}\right) + C \\
&= \frac{\theta}{48} + \frac{\sin 2\theta}{96} + C
\end{aligned}$$

Now we have to write this answer in terms of x. Since $x = (2/3)\tan\theta$, therefore, $\theta = \tan^{-1}(3x/2)$. To find $\sin 2\theta$ in terms of x, we have to complete the right triangle as below. Since $3x/2 = \tan\theta$, therefore the opposite is $3x$ and the adjacent is 2. The hypotenuse is therefore $\sqrt{4+9x^2}$.



From the picture above,

$$\cos\theta = \frac{2}{\sqrt{4+9x^2}}$$

and

$$\sin\theta = \frac{3x}{\sqrt{4+9x^2}}$$

Now

$$\sin 2\theta = 2\sin\theta\cos\theta = 2\cdot\frac{2}{\sqrt{4+9x^2}}\cdot\frac{3x}{\sqrt{4+9x^2}} = \frac{12x}{4+9x^2}$$

Therefore,

$$\begin{aligned}
\int \frac{dx}{(4+9x^2)^2} &= \frac{\theta}{48} + \frac{\sin 2\theta}{96} + C \\
&= \frac{1}{48}\tan^{-1}(\frac{3x}{2}) + \frac{1}{96}\cdot\frac{12x}{4+9x^2} + C \\
&= \frac{1}{48}\tan^{-1}(\frac{3x}{2}) + \frac{x}{8(4+9x^2)} + C
\end{aligned}$$

Going back to the original definite integral we have,

$$\begin{aligned}
\int_0^1 \frac{dx}{(4+9x^2)^2} &= \left[\frac{1}{48}\tan^{-1}(\frac{3x}{2}) + \frac{x}{8(4+9x^2)}\right]\Bigg|_0^1 \\
&= \frac{1}{48}\tan^{-1}\left(\frac{3}{2}\right) + \frac{1}{104} - (0+0) \\
&= \frac{1}{48}\tan^{-1}\left(\frac{3}{2}\right) + \frac{1}{104}
\end{aligned}$$

$\square$

Form 3. Function involving $x^2 - a^2$. When we have a function that involves this form then we use the substitution

$$x = a\sec\theta$$

and we continue with the integration. We look at an example illustrating this substitution.

Example 8.3. Evaluate

$$\int \frac{1}{\sqrt{x^2-9}}\,dx$$

Solution. We see that we have the form $x^2 - a^2$ in the expression, with $a = 3$. Hence, we use the substitution,

$$x = 3\sec\theta$$

Therefore,

$$dx = 3\sec\theta\tan\theta\,d\theta$$

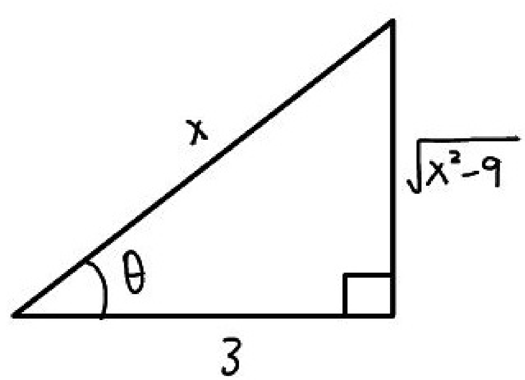
Substituting back we have,

$$\begin{aligned}
\int \frac{1}{\sqrt{x^2-9}}\,dx &= \int \frac{3\sec\theta\tan\theta\,d\theta}{\sqrt{9\sec^2\theta-9}} \\
&= \int \frac{3\sec\theta\tan\theta d\theta}{\sqrt{9\tan^2\theta}} \\
&= \int \frac{3\sec\theta\tan\theta d\theta}{3\tan\theta} \\
&= \int \sec\theta\,d\theta
\end{aligned}$$

This is a standard integral.

$$\int \sec\theta\,d\theta = \ln|\sec\theta+\tan\theta| + C$$

Now we have to write this answer in terms of x. Since $x = 3\sec\theta$, therefore, $\theta = \sec^{-1}(x/3)$. To find $\tan\theta$ in terms of x, we have to complete the right triangle as below. Since $x/3 = \sec\theta$, therefore the adjacent is 3 and the hypotenuse is x. The opposite is therefore $\sqrt{x^2-9}$.



From the picture above,

$$\tan\theta = \frac{\sqrt{x^2-9}}{3}$$

Therefore,

$$\begin{aligned}
\int \frac{1}{\sqrt{x^2-9}}\,dx &= \ln|\sec\theta + \tan\theta| + C \\
&= \ln\left|\frac{x}{3} + \frac{\sqrt{x^2-9}}{3}\right| + C \\
&= \ln\left|\frac{x+\sqrt{x^2-9}}{3}\right| + C \\
&= \ln\left|x+\sqrt{x^2-9}\right| - \ln 3 + C
\end{aligned}$$

Since $\ln 3$ is also a constant so $-\ln 3 + C$ is also a constant, hence,

$$\int \frac{1}{\sqrt{x^2-9}}\,dx = \ln\left|x+\sqrt{x^2-9}\right| + C$$

□

8.2. The Method of Partial Fractions

8.2.1. Performance Criteria

(a) Evaluate an integral using partial fractions, for a linear and quadratic denominator with or without repeated factors.

8.2.2. Partial Fractions

In this section we learn a method by which we can integrate rational functions of the form

$$f(x) = \frac{P(x)}{Q(x)}$$

When we have a rational function like this, we want to make sure that

$$\text{degree } P(x) < \text{degree } Q(x)$$

If not, then we perform long division to find the quotient $Z(x)$ and remainder $R(x)$, and then we can write,

$$f(x) = \frac{P(x)}{Q(x)} = Z(x) + \frac{R(x)}{Q(x)}$$

Now we work with

$$\frac{R(x)}{Q(x)}$$

since

$$\text{degree } R(x) < \text{degree } Q(x)$$

The idea here is to write $f(x)$ as a sum of simpler rational functions that can be integrated easily.

Strategy. We look at $Q(x)$ and factor it if possible. We look at 2 cases.

Case 1. $Q(x)$ is a product of non repeated factors of the form $ax + b$ or $ax^2 + bx + c$ or both - In this case,

$$\frac{P(x)}{Q(x)} = \frac{P(x)}{(a_1x + b_1)\cdots(a_nx + b_n)\cdots(A_1x^2 + B_1x + C_1)\cdots(A_2x^2 + B_2x + C_2)}$$

We can then use **partial fraction decomposition** to write

$$\frac{P(x)}{Q(x)} = \frac{D_1}{a_1x + b_1} + \cdots + \frac{D_n}{a_nx_n + b_n} + \frac{E_1x + F_1}{A_1x^2 + B_1x + C_1} + \cdots + \frac{E_2x + F_2}{A_2x^2 + B_2x + C_2}$$

Note that the numerator of the fraction with a linear term (degree=1) in the denominator is a constant (degree=0) and the numerator of the fraction with a quadratic term (degree=2) in the denominator is a linear term (degree=1). This is the strategy in general. If there is a polynomial of degree n in the denominator then the corresponding term in the numerator will be a general polynomial of degree $n-1$. We then use algebra to find the constants $D_1, \ldots, D_n, E_1, \ldots, E_n, F_1, \ldots, F_n$.

Example 8.4. Evaluate

$$\int \frac{3}{(x-1)(x^2+x)}\,dx$$

Solution. As x^2+x can be factored, therefore we write the rational function as

$$\int \frac{3}{(x-1)(x^2+x)}\,dx = \int \frac{3}{(x-1)x(x+1)}\,dx$$

We decompose it as

$$\frac{3}{(x-1)x(x+1)} = \frac{A}{x-1} + \frac{B}{x} + \frac{C}{x+1}$$

We multiply both sides by $(x-1)x(x+1)$ to get

$$3 = Ax(x+1) + B(x-1)(x+1) + Cx(x-1)$$

Substituting $x = 0$ we have,

$$\begin{aligned} 3 &= A \cdot 0 \cdot 1 + B \cdot (-1) \cdot 1 + C \cdot 0 \cdot (-1) \\ 3 &= -B \\ B &= -3 \end{aligned}$$

Substituting $x = 1$ we have,

$$\begin{aligned} 3 &= A \cdot 1 \cdot 2 + B \cdot 0 \cdot 2 + C \cdot 1 \cdot 0 \\ 3 &= 2A \\ A &= \frac{3}{2} \end{aligned}$$

Substituting $x = -1$ we have,

$$\begin{aligned} 3 &= A \cdot (-1) \cdot 0 + B \cdot (-2) \cdot 0 + C \cdot (-1) \cdot (-2) \\ 3 &= 2C \\ C &= \frac{3}{2} \end{aligned}$$

Therefore we rewrite the integral as

$$\begin{aligned} \int \frac{3}{(x-1)(x^2+x)}\,dx &= \int \frac{3/2}{x-1}\,dx + \int \frac{-3}{x}\,dx + \int \frac{3/2}{x+1}\,dx \\ &= \frac{3}{2}\ln|x-1| - 3\ln|x| + \frac{3}{2}\ln|x+1| + C \end{aligned}$$

□

Case 2. **$Q(x)$ is a product of repeated factors of the form $ax+b$ or ax^2+bx+c or both** - In this case let us look at a rational function of the form

$$\frac{P(x)}{Q(x)} = \frac{P(x)}{(a_1x+b_1)^r(A_1x^2+B_1x+C_1)^s}$$

We can then use **partial fraction decomposition** to write

$$\frac{P(x)}{Q(x)} = \frac{D_1}{a_1x+b_1}+\cdots+\frac{D_r}{(a_1x+b_1)^r}+\frac{E_1x+F_1}{A_1x^2+B_1x+C_1}+\cdots+\frac{E_sx+F_s}{(A_1x^2+B_1x+C_1)^s}$$

We see here that for every repeated factor in $Q(x)$, we have a separate rational function following the same technique as Case 1. We then use algebra to find the constants $D_1,\ldots,D_r,E_1,\ldots,E_s,F_1,\ldots,F_s$. Let us look at an example.

Example 8.5. Evaluate

$$\int \frac{x^2}{(x+1)(x^2+1)}\,dx$$

Solution. Following the strategy from above, we decompose the expression as

$$\frac{x^2}{(x+1)(x^2+1)} = \frac{A}{x+1}+\frac{Bx+C}{(x^2+1)}$$

Multiplying both sides by $(x+1)(x^2+1)$ we get,

$$x^2 = A(x^2+1)+(Bx+C)(x+1) \tag{8.6}$$

Substituting $x=-1$ we get,

$$\begin{aligned} 1 &= A\cdot 2+(-B+C)\cdot 0 \\ 1 &= 2A \\ A &= \frac{1}{2} \end{aligned}$$

To find B and C, we foil the right side of Equation 8.6, to get

$$\begin{aligned} x^2 &= Ax^2+A+Bx^2+Bx+Cx+C \\ x^2 &= (A+B)x^2+(B+C)x+(A+C) \end{aligned}$$

Equating coefficients of x^2, x and the constant term we have.

$$A+B=1 \Rightarrow B=1-A=\frac{1}{2}$$

$$B+C=0 \Rightarrow C=-B=-\frac{1}{2}$$

Therefore, we can rewrite the integral as

$$\begin{aligned}
\int \frac{x^2}{(x+1)(x^2+1)}\,dx &= \int \frac{\frac{1}{2}}{x+1}\,dx + \frac{\frac{1}{2}x-\frac{1}{2}}{x^2+1}\,dx \\
&= \frac{1}{2}\int \frac{dx}{x+1} + \frac{1}{2}\int \frac{x-1}{x^2+1}\,dx \\
&= \frac{1}{2}\ln|x+1| + \frac{1}{2}\int \frac{x-1}{x^2+1}\,dx
\end{aligned}$$

To evaluate $\frac{1}{2}\int \frac{x-1}{x^2+1}\,dx$, we do some manipulations,

$$\begin{aligned}
\frac{1}{2}\int \frac{x-1}{x^2+1}\,dx &= \frac{1}{4}\int \frac{2x-2}{x^2+1}\,dx \\
&= \frac{1}{4}\int \frac{2x}{x^2+1}\,dx - \frac{1}{2}\int \frac{1}{x^2+1}\,dx \\
&= \frac{1}{4}\int \frac{2x}{x^2+1}\,dx - \frac{1}{2}\tan^{-1}x
\end{aligned}$$

We can use the substitution $u = x^2+1$ for the first integral. Then $du = 2x\,dx$, therefore,

$$\begin{aligned}
\frac{1}{4}\int \frac{2x}{x^2+1}\,dx &= \frac{1}{4}\int \frac{du}{u} \\
&= \frac{1}{4}\ln|u| + C = \frac{1}{4}\ln|x^2+1| + C
\end{aligned}$$

Therefore,

$$\frac{1}{2}\int \frac{x-1}{x^2+1}\,dx = \frac{1}{4}\ln|x^2+1| - \frac{1}{2}\tan^{-1}x + C$$

Hence the original integral becomes,

$$\begin{aligned}
\int \frac{x^2}{(x+1)(x^2+1)}\,dx &= \frac{1}{2}\ln|x+1| + \frac{1}{2}\int \frac{x-1}{x^2+1}\,dx \\
&= \frac{1}{2}\ln|x+1| + \frac{1}{4}\ln|x^2+1| - \frac{1}{2}\tan^{-1}x + C
\end{aligned}$$

□

Handout 5
MATH 252

Dibyajyoti Deb

5.1. Setting up Integral: Volume, Density, Average Values

5.1.1. Performance Criteria

(a) Use integration to find the volume of solids given the cross section and the base.
(b) Use integration to find the mass of a solid with variable density.
(c) Compute the average value of an integrable function $f(x)$ on $[a, b]$.

5.1.2. Setting up integral

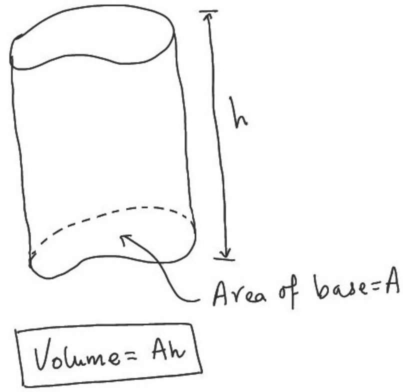
In this section we look at various applications of the integral. We start off by finding the volume of a solid given the base and the cross section.

5.1.2.1. **Volume.** From previous knowledge we know that the volume V, of a right circular cylinder is

$$V = \pi r^2 h$$

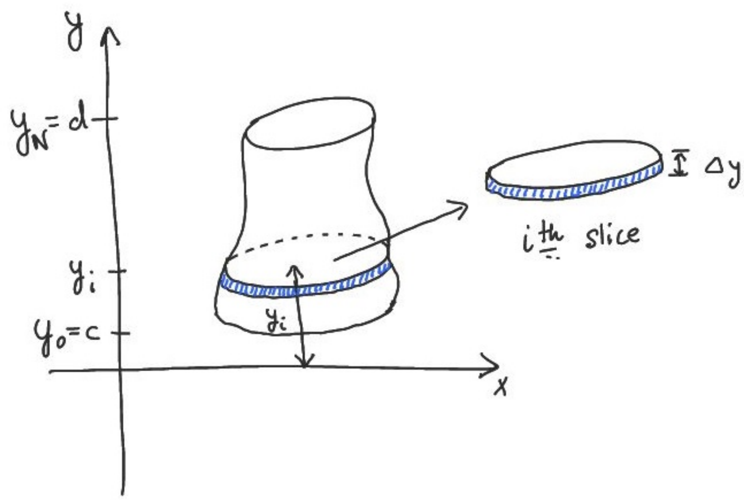
where r is the radius of the base and h is the height perpendicular to the base. Here πr^2 is the area of the base (which is a circle of radius r) or any horizontal cross sections (which are all circles of radius r).

In fact as long as the sides are perpendicular to the base, the volume of a *right cylinder* is Ah where A is the area of the base which need not be circular and h is the height measured perpendicular to the base.



Now, can we extend this method to find volume of solids which are more general? Yes, we can as long as we find the area of an arbitrary cross section. Let's look at this in more detail.

We divide the solid into N thin horizontal slices. Each slice is so small in width that it resembles a right cylinder in itself.



Each of these horizontal slices have thickness $\Delta y = (b-a)/N$. Let the area of the ith slice be $A(y_i)$ which is at a distance of y_i units from the x-axis. Since each slice resembles a right cylinder, therefore the volume of the i slice is

$$V_i = A(y_i)\Delta y$$

Thus the total volume of all the N slices put together would be

$$V = \sum_{i=1}^{N} V_i \approx \sum_{i=1}^{N} A(y_i)\Delta y$$

This is an approximation of the volume of the solid. To find the exact volume we would need to make infinite number of slices which would mean $N \to \infty$. In that case the volume ends up becoming,

$$V = \lim_{N\to\infty} \sum_{i=1}^{N} A(y_i)\Delta y = \int_c^d A(y)\, dy$$

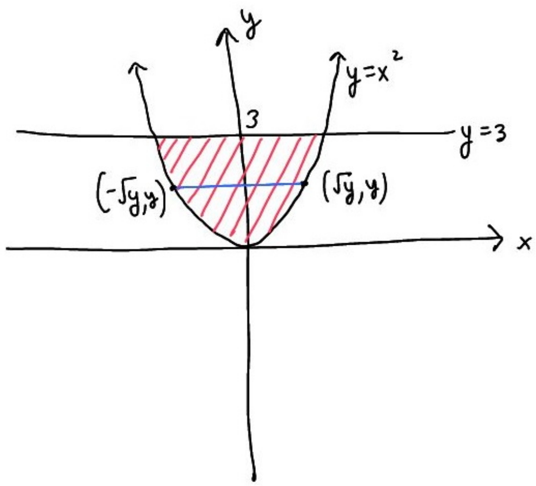
by the definition of definite integral. Therefore,

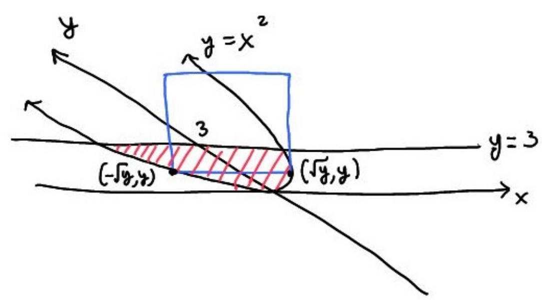
Fact. Let $A(y)$ be the area of a horizontal cross section at a distance y from the x-axis of a solid body extending from $y = c$ to $y = d$. Then,

$$\text{Volume of the solid body} = \int_c^d A(y)\, dy \tag{5.1}$$

Example 5.2. Find the volume of the solid whose base is the region enclosed by $y = x^2$ and $y = 3$ and the cross sections perpendicular to the y-axis are squares.

Solution. Let us first draw the base and the cross section so that we can better visualize the solid.





The shaded region in red above represents the base of the solid. We draw a cross section which is a square in blue at a distance of y units from the x-axis. This cross section touches the curve $y = x^2$ at the points $A(\sqrt{y}, y)$ and $B(-\sqrt{y}, y)$. Hence, the length of the side of the square is $2\sqrt{y}$. Therefore,

$$A(y) = (2\sqrt{y})^2 = 4y.$$

The limits of the definite integral are from 0 to 3 as those are the extremities of the base on the y-axis. Hence,

$$\begin{aligned} V &= \int_0^3 A(y)\,dy \\ &= \int_0^3 4y\,dy \\ &= 2y^2\Big|_0^3 = 18 \text{ cubic units.} \end{aligned}$$

□

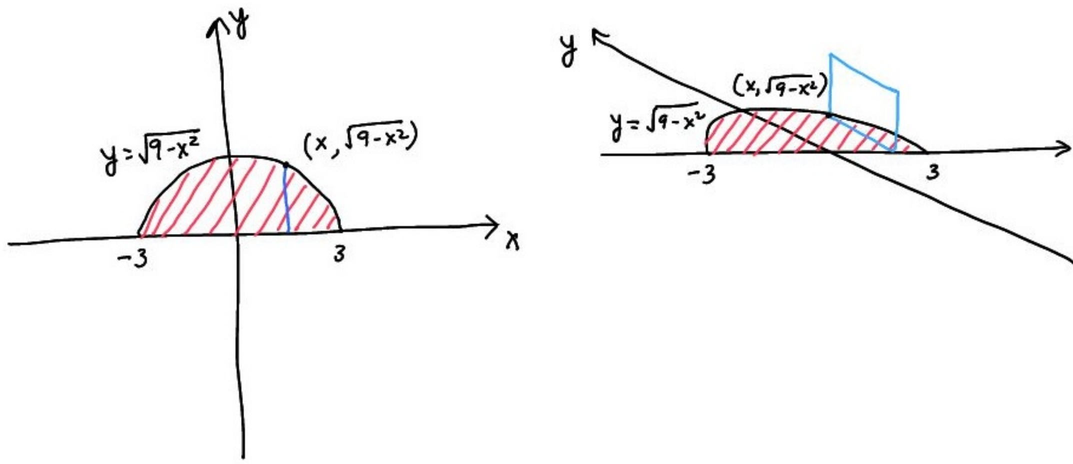
We have seen, what we need to do when we use horizontal cross sections perpendicular to the y-axis. What if we want to use vertical cross sections perpendicular to the x-axis. We can use the same technique as above except that now we have to find the area of an arbitrary vertical cross section in terms of x. This cross section is at a distance of x units from the y-axis. Therefore,

Fact. Let $A(x)$ be the area of a vertical cross section at a distance x from the y-axis of a solid body extending from $x = a$ to $x = b$. Then,

$$\boxed{\text{Volume of the solid body } = \int_a^b A(x)\,dx} \tag{5.3}$$

Example 5.4. Find the volume of the solid whose base is the semicircle $y = \sqrt{9 - x^2}$, where $-3 \leq x \leq 3$ and the cross sections perpendicular to the x-axis are squares.

Solution. Let us first draw the base and the cross section so that we can better visualize the solid.



The shaded region in red above represents the base of the solid. We draw a cross section which is a square in blue at a distance of x units from the y-axis. This cross section touches the semicircle $y = \sqrt{9-x^2}$ at the point $(x, \sqrt{9-x^2})$. Hence, the length of the side of the square is $\sqrt{9-x^2}$. Therefore,

$$A(x) = (\sqrt{9-x^2})^2 = 9 - x^2.$$

The limits of the definite integral are from -3 to 3 as those are the extremities of the base on the x-axis. Hence,

$$\begin{aligned} V &= \int_{-3}^{3} A(x)\,dx \\ &= \int_{-3}^{3} (9-x^2)\,dx \\ &= 9x - \frac{x^3}{3}\Big|_{-3}^{3} \\ &= (27 - \frac{27}{3}) - (-27 - (\frac{-27}{3})) = 36 \text{ cubic units.} \end{aligned}$$

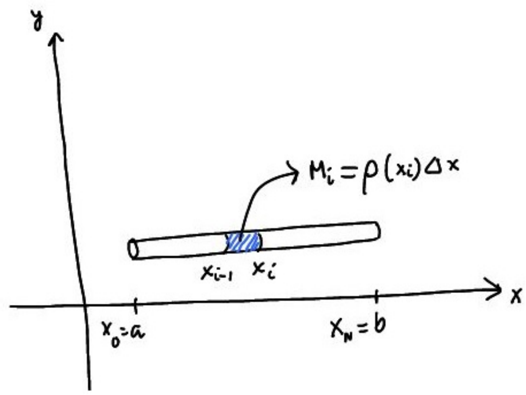
□

5.1.2.2. **Density.** In this section we find the mass of a rod whose density is variable. This is called **linear mass density** ρ and is defined as the mass per unit length. If ρ is constant, then by definition,

$$\text{Mass} = \text{linear mass density} \times \text{length} = \rho \cdot l$$

Now if we have a rod extending along the x-axis, and its density at a point x varies according to the density function $\rho(x)$ then we find the mass of the rod by using integration and the same technique that we used before in computing areas and volume.

We divide the rod into N equal segments. Each segment is so small such that $\rho(x)$ is a constant along the ith segment.



Each of these segments have thickness $\Delta x = (b-a)/N$. Let the mass of the ith segment be M_i. Therefore mass of the ith slice is

$$M_i = \rho(x_i)\Delta x$$

Thus the total mass of all the N segments put together would be

$$M = \sum_{i=1}^{N} M_i \approx \sum_{i=1}^{N} \rho(x_i)\Delta x$$

This is an approximation of the mass of the rod. To find the exact mass we would need to make infinite number of segments which would mean $N \to \infty$. In that case the mass ends up becoming,

$$M = \lim_{N\to\infty} \sum_{i=1}^{N} \rho(x_i)\Delta x = \int_a^b \rho(x)\,dx$$

by the definition of definite integral. Therefore,

Fact. Let $\rho(x)$ be the density function of a rod along the x-axis extending from $x = a$ to $x = b$. Then,

$$\boxed{\text{Mass of the rod } = \int_a^b \rho(x)\,dx}$$

Example 5.5. Find the total mass of a 1 m rod whose linear density function is $\rho(x) = 10(x+1)^{-2}$ kg/m for $0 \le x \le 1$.

Solution. Applying the result that we just derived,

$$\text{Mass of the rod } = \int_0^1 \frac{10}{(x+1)^2}\,dx$$

We use substitution to evaluate the above definite integral.

$$\begin{aligned} u &= x+1 \\ du &= dx \end{aligned}$$

Changing the limits,

$$\text{When } x = 0, u = 0+1 = 1.$$

$$\text{When } x = 1, u = 1+1 = 2.$$

Thus,

$$\begin{aligned} \text{Mass of the rod} &= \int_1^2 \frac{10}{u^2}\,du \\ &= -\frac{10}{u}\bigg|_1^2 \\ &= (-\frac{10}{2}) - (-\frac{10}{1}) = 5 \text{ kg.} \end{aligned}$$

□

5.1.3. Average Values

Given a set of students and their exam scores, we know how to find the average score. We can do this because the number of students in the class is finite number and hence a discreet set. What if we want to find out the average value of a function $f(x)$ on a continuous interval $[a, b]$?

This is not an easy task since there are infinite number of points on the interval $[a, b]$. Let's say we divide the interval $[a, b]$ into N equal sub intervals at the points $x_1, x_2, \ldots, x_N$. In this case $\Delta x = (b-a)/N$. Using the right end point approximation

$$\begin{aligned} R_N &= (f(x_1) + f(x_2) + \cdots + f(x_N))\Delta x \\ &= (f(x_1) + f(x_2) + \cdots + f(x_N))\Big(\frac{b-a}{N}\Big) \\ \Big(\frac{1}{b-a}\Big)R_N &= \frac{f(x_1) + f(x_2) + \cdots + f(x_N)}{N} \end{aligned}$$

Therefore, $\Big(\frac{1}{b-a}\Big)R_N$ is the average value of the function $f(x)$ with points $x_1, \ldots, x_N$. But what about the remaining infinite points? If we take the limit as $N \to \infty$ we have,

$$\begin{aligned}
\lim_{N\to\infty} \left(\frac{1}{b-a}\right) R_N &= \lim_{N\to\infty} \frac{f(x_1) + f(x_2) + \cdots + f(x_N)}{N} \\
\left(\frac{1}{b-a}\right) \lim_{N\to\infty} R_N &= \text{Average value of } f(x) \text{ on } [a,b]
\end{aligned}$$

Now from previous knowledge, the right end point approximation approaches the definite integral as the number of points approaches infinity. Hence,

$$\lim_{N\to\infty} R_N = \int_a^b f(x)\,dx$$

Therefore,

$$\boxed{\text{Average value of } f(x) \text{ on } [a,b] = \left(\frac{1}{b-a}\right) \int_a^b f(x)\,dx}$$

Example 5.6. Calculate the average value of the function $f(x)$ on the interval $[-1, 1]$.

$$f(x) = x^3$$

Solution. Here $a = -1$ and $b = 1$, therefore using the formula for the average value from above we have,

$$\begin{aligned}
\text{Average value of } f(x) \text{ on } [-1,1] &= \left(\frac{1}{1-(-1)}\right) \int_{-1}^{1} x^3\,dx \\
&= \frac{1}{2}\left(\frac{x^4}{4}\right)\Bigg|_{-1}^{1} = 0
\end{aligned}$$

□

5.1.3.1. **Mean Value Theorem for Integrals.** Now we look at an application of the average value of a function over the continuous interval $[a, b]$. Going back to the previous example of student scores, suppose that the class consists of 3 students whose scores are 7, 10, and 3 out of 10. The average of these scores are 6.67, however, no one in the class got this as their score. This is because the set of students is a discrete set as we discussed earlier, however, this is not case when we look at a function on an interval $[a, b]$.

Theorem 5.7. Mean Value Theorem for Integrals *If $f(x)$ is continuous of $[a, b]$, then there exists a value $c \in [a, b]$ such that*

$$f(c) = \left(\frac{1}{b-a}\right) \int_a^b f(x)\,dx$$

Hence, the above theorem says that there will exist a value c on the interval $[a, b]$ where the value of the function $f(x)$ is equal to the average value of the function on that interval.

Example 5.8. Let $f(x) = \sqrt{x}$. Find a value of c in $[4, 9]$ such that $f(c)$ is equal to the average of f on $[4, 9]$.

Solution. Let us first calculate the average value of $f(x)$ on $[4, 9]$.

$$\begin{aligned}
\left(\frac{1}{b-a}\right) \int_a^b f(x)\,dx &= \frac{1}{5} \int_4^9 \sqrt{x}\,dx \\
&= \frac{1}{5}\left(\frac{2x^{3/2}}{3}\right)\Bigg|_4^9 \\
&= \frac{1}{5}\left(\frac{2 \cdot 9^{3/2}}{3} - \frac{2 \cdot 4^{3/2}}{3}\right) \\
&= \frac{1}{5}\left(18 - \frac{16}{3}\right) = \frac{38}{15}.
\end{aligned}$$

Now we look for c, such that

$$\begin{aligned}
f(c) &= \frac{38}{15} \\
\sqrt{c} &= \frac{38}{15} \\
c &= \left(\frac{38}{15}\right)^2 = \frac{1444}{225} = 6.418.
\end{aligned}$$

□

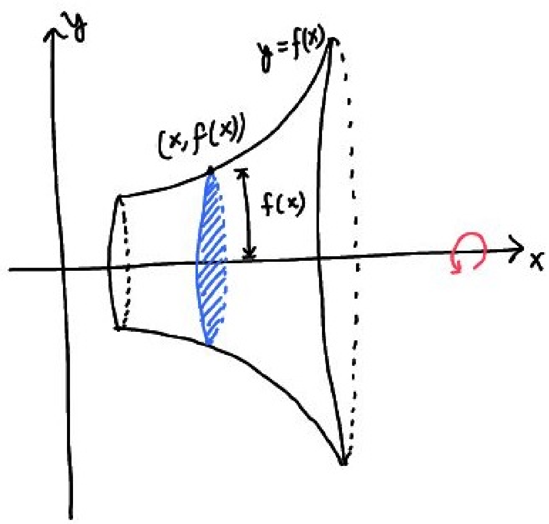
5.2. Volumes of Revolution

5.2.1. Performance Criteria

(a) Set up an integral representing the volume of a solid of revolution about a coordinate axis, given the formulas for solids of revolution.

5.2.2. Volumes of revolution about the x-axis

A **solid of revolution** is a solid obtained by rotating a region in the plane about an axis. We first look at what happens when we rotate the region along the x-axis.



As we rotate the region under the curve $y = f(x)$, we end up with a solid whose vertical cross sections perpendicular to the x-axis at a distance of x units from the y-axis are disks of radius $R = f(x)$. By (5.3), we have,

$$\begin{aligned} V &= \int_a^b A(x)\,dx \\ &= \int_a^b \pi R^2\,dx \\ &= \pi \int_a^b f(x)^2\,dx \end{aligned}$$

Thus,

Theorem 5.9. Volume of revolution about x-axis: Disk method *If $f(x)$ is continuous and $f(x) \geq 0$ on $[a, b]$, then the solid obtained by rotating the region under the graph about the x-axis has volume*

$$V = \pi \int_a^b f(x)^2\,dx \tag{5.10}$$

Example 5.11. Find the volume of the solid obtained by rotating the curve $y = x^3$ on $[1, 3]$ about the x-axis.

Solution. By (5.13), we have

$$\begin{aligned} V &= \pi \int_1^3 \frac{(x^3)^2}{2}\,dx \\ &= \frac{\pi}{2} \int_1^3 x^6\,dx \\ &= \frac{\pi}{2} \cdot \frac{x^7}{7} \bigg|_1^3 \\ &= \frac{\pi}{2}\left(\frac{3^7}{7} - \frac{1^7}{7}\right) = \frac{2186}{14}\pi. \end{aligned}$$

□

5.2.3. Volumes of revolution about the y-axis

Just as in rotation around the x-axis, we end up with horizontal cross sections which are disks when we rotate the region to the left of the curve $x = f(y)$ around the y-axis. If this cross section is at a distance of y units from the x axis, then the radius of this disk is $R = f(y)$. By (5.1), we have,

Theorem 5.12. Volume of revolution about y-axis: Disk method *If $f(y)$ is continuous and $f(y) \geq 0$ on $[c, d]$, then the solid obtained by rotating the region left of the graph about the y-axis has volume*

$$V = \pi \int_c^d f(y)^2\,dy \tag{5.13}$$

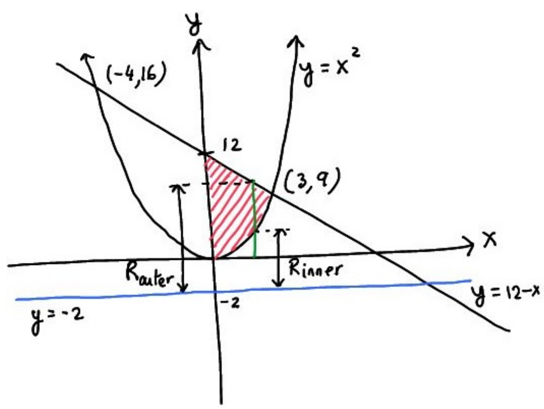
5.2.4. Volume of revolution about a horizontal line $y = c$ of a region between two curves

If we have two curves $y = f(x)$ and $y = g(x)$ with $f(x) \geq g(x)$ on $[a, b]$ and we rotate the region between these two curves around a horizontal line $y = c$, then we obtain a solid which has a "hole" in between. Any vertical cross sections perpendicular to the x-axis results in a washer rather than a disk. This washer is made up of two concentric disks of radius R_{outer} and R_{inner}. These radiuses are calculated from the horizontal line $y = c$. Hence the volume of the solid is,

$$\begin{aligned} V &= \int_a^b A(x)\,dx \\ &= \int_a^b (\pi R_{\text{outer}}^2 - \pi R_{\text{inner}}^2)\,dx \\ &= \pi \int_a^b (R_{\text{outer}}^2 - R_{\text{inner}}^2)\,dx \end{aligned}$$

Example 5.14. Find the volume of the solid obtained by rotating the region enclosed by the graphs $y = x^2$, $y = 12 - x$ and $x = 0$ about the horizontal line $y = -2$.

Solution. We first draw a picture of the region between the curves.



The point of intersection of the curves $y = x^2$ and $y = 12 - x$ can be found by

$$\begin{aligned} x^2 &= 12 - x \\ x^2 + x - 12 &= 0 \\ (x+4)(x-3) &= 0 \end{aligned}$$

$$x = -4 \text{ and } x = 3$$

Since the region is bounded by $x = 0$, therefore the point of intersection is at $(3, 9)$.

When we take a vertical cross section perpendicular to the x-axis, then

$$R_{\text{outer}} = (12 - x) + 2 = 14 - x$$

and

$$R_{\text{inner}} = (x^2) + 2$$

Thus volume of the solid obtained from rotation about the line $y = -2$ is,

$$\begin{aligned} V &= \pi \int_0^3 ((14-x)^2 - (x^2+2)^2)\,dx \\ &= \pi \int_0^3 (196 - 28x + x^2 - x^4 - 4x^2 - 4)\,dx \\ &= \pi \int_0^3 (192 - 28x - 3x^2 - x^4)\,dx \\ &= \pi(192x - 14x^2 - x^3 - \frac{x^5}{5})\Big|_0^3 \\ &= \pi(192 \cdot 3 - 14 \cdot (3)^2 - (3)^3 - \frac{3^5}{5}) = \frac{1872\pi}{5} \end{aligned}$$

□

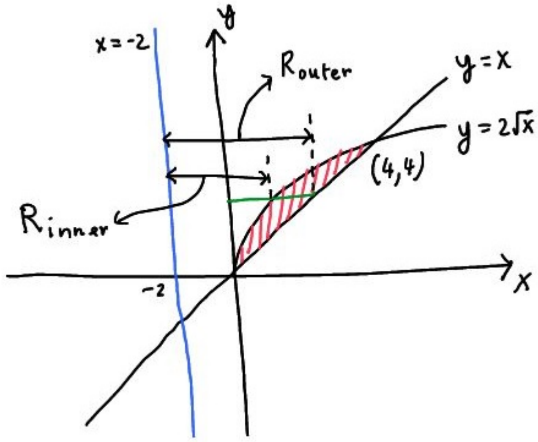
5.2.5. Volume of revolution about a vertical line $x = c$ of a region between two curves

Here again if we have two curves $x = f(y)$ and $x = g(y)$ with $f(y) \geq g(y)$ on $[c, d]$ and we rotate the region between these two curves around a vertical line $x = e$, then we obtain a solid which has a "hole" in between. Any horizontal cross sections perpendicular to the y-axis results in a washer rather than a disk. This washer is made up of two concentric disks of radius R_{outer} and R_{inner}. These radiuses are calculated from the vertical line $x = e$. Hence the volume of the solid is,

$$\begin{aligned} V &= \int_c^d A(y)\,dy \\ &= \int_c^d \pi(R_{\text{outer}}^2 - \pi R_{\text{inner}}^2)\,dy \\ &= \pi \int_c^d (R_{\text{outer}}^2 - R_{\text{inner}}^2)\,dy \end{aligned}$$

Example 5.15. Find the volume of the solid obtained by rotating the region enclosed by the graphs $y = 2\sqrt{x}$ and $y = x$ about the vertical line $x = -2$.

Solution. We first draw a picture of the region between the curves.



We find the points of intersection of the curves $y = 2\sqrt{x}$ and $y = x$ by,

$$\begin{aligned} 2\sqrt{x} &= x \\ 4x &= x^2 \\ x(x-4) &= 0 \end{aligned}$$

$$x = 4 \text{ and } x = 0$$

Thus, the points of intersection are $(4, 4)$ and $(0, 0)$. We now write the equations of the curves as functions of y. Since $y = 2\sqrt{x}$, therefore $x = \dfrac{y^2}{4}$ for $y \geq 0$ and $y = x$ implies that $x = y$.

When we take a horizontal cross section perpendicular to the y-axis, then

$$R_{\text{outer}} = (y) + 2$$

and

$$R_{\text{inner}} = (\frac{y^2}{4}) + 2$$

Thus the volume of the solid obtained from rotation about the line $x = -2$ is,

$$\begin{aligned} V &= \pi \int_0^4 ((y+2)^2 - (\frac{y^2}{4} + 2)^2)\, dy \\ &= \pi \int_0^4 (y^2 + 4y + 4 - \frac{y^4}{16} - y^2 - 4)\, dy \\ &= \pi \int_0^4 (4y - \frac{y^4}{16})\, dy \\ &= \pi(2y^2 - \frac{y^5}{80})\Big|_0^4 = \frac{96\pi}{5} \end{aligned}$$

$\square$

Handout 6
MATH 252

Dibyajyoti Deb

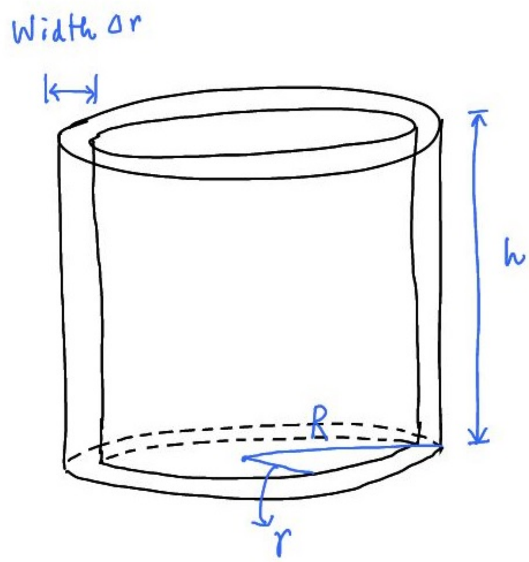
6.1. The Method of Cylindrical Shells

6.1.1. Performance Criteria

(a) Set up an integral using the method of cylindrical shells to represent the volume of a solid of revolution about a coordinate axis.

6.1.2. The Shell Method

In this section we learn a different method by which we can compute the volume of a solid of revolution. This method is based on cylindrical shells and is more convenient in some cases. We first look at a thin cylindrical shell of height h and approximate its volume.



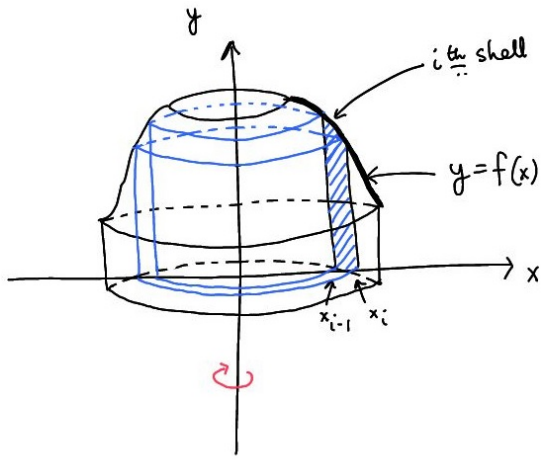
If the radius of the outer cylinder is R and the radius of the inner cylinder is r, then the volume of the region in between these two cylinders is given by

$$\pi R^2 h - \pi r^2 h = \pi h(R^2 - r^2) = \pi h(R+r)(R-r)$$

Now since the shell is very thin therefore $r \approx R$, so that we can consider the radius of the shell itself to be R, and hence $R + r \approx 2R$ and the thickness of the shell is given by $R - r = \Delta r$. Thus,

$$\text{Volume of the shell } \approx 2\pi Rh\Delta r = 2\pi(\text{radius})(\text{height of the shell})(\text{thickness})$$

Now if we take the region under the curve $y = f(x)$ on $[a, b]$ and rotate it about the y-axis, then we end up with a solid that can be divided into thin concentric shells. Each of these shells are formed from a thin strip on the x-axis of width Δx as shown in the figure below.



If the volume of the thin shell that is formed by rotating the strip on $[x_{i-1}, x_i]$ is V_i, then the radius of the shell is x_i and the height is $f(x_i)$. Therefore,

$$V_i \approx 2\pi(\text{radius}) \times (\text{height of the shell}) \times (\text{thickness}) = 2\pi x_i f(x_i)\Delta x$$

Since the region under $y = f(x)$ is made up N such strips therefore, the approximate volume of the shell is

$$V = \sum_{i=1}^{N} V_i = 2\pi \sum_{i=1}^{N} x_i f(x_i)\Delta x$$

We find the exact volume if we take thinner and more number of strips i.e. $N \to \infty$. In that case the sum on the right converges to $2\pi \int_a^b x f(x)\, dx$. Thus,

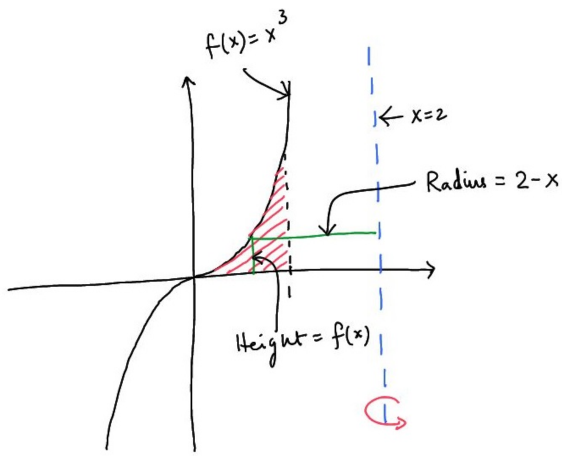
Theorem 6.1. Volume of Revolution: The Shell Method *The solid obtained by rotating the region under $y = f(x)$ over the interval $[a, b]$ about the y-axis has volume*

$$V = 2\pi \int_a^b x f(x)\, dx = 2\pi \int_a^b (\textit{radius})(\textit{height of the shell})\, dx$$

Now let us look at an example which uses the same theory from above but where the region is rotated about a vertical axis other than the y-axis.

Example 6.2. Find the volume of the solid obtained by rotating the region underneath the graph of $y = x^3$ on $[0, 1]$ about the line $x = 2$.

Solution. Let us first sketch the graph and the region underneath it.

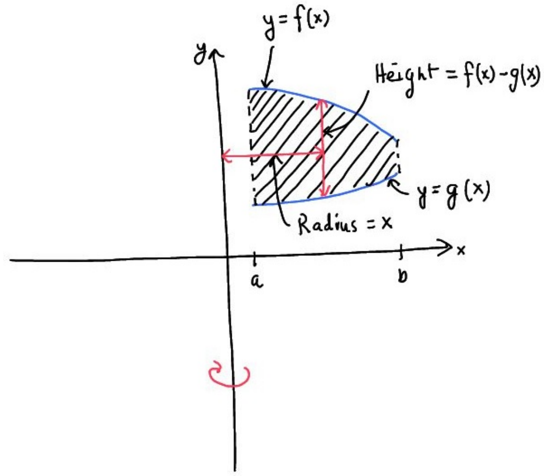


The radius of the cylindrical shell is the distance of the shell from the axis of rotation. At any point $(x, f(x))$ on the curve the cylindrical shell will have radius $2 - x$ (the distance of the point from $x = 2$) and height $f(x) = x^3$. Thus using Theorem 6.1, we have

$$\begin{aligned} V &= 2\pi \int_0^1 (2 - x)x^3\, dx \\ &= 2\pi \int_0^1 (2x^3 - x^4)\, dx \\ &= 2\pi \left(\frac{x^4}{2} - \frac{x^5}{5}\right)\Bigg|_0^1 = \frac{3\pi}{5} \end{aligned}$$

□

6.1.2.1. **Shell Method : Volume of revolution about a vertical axis of a region between two curves.** What happens when we rotate the region between two curves around a vertical axis? The theory still remains the same! We still have to find the radius of a cylindrical shell and its height.

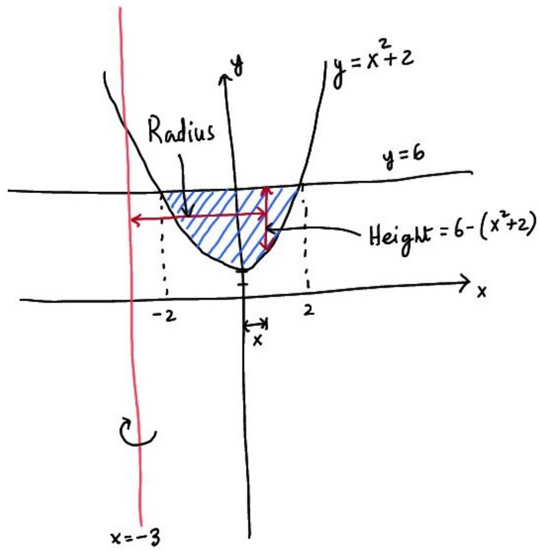


Looking at the picture above we see that the height of the shell between the curves $y = f(x)$ and $y = g(x)$ is $f(x) - g(x)$ and the radius of the shell is still x (its distance from the axis of rotation which in this case is the y-axis). Thus, the volume of the solid of revolution from this region is given by,

$$\boxed{V = 2\pi \int_a^b (\text{radius})(\text{height of the shell})\, dx = 2\pi \int_a^b x(f(x) - g(x))\, dx}$$

Example 6.3. Use the Shell Method to find the volume obtained by rotating the region between the curves $y = x^2 + 2$, $y = 6$ and $x \geq 0$ about the vertical axis $x = -3$.

Solution. As always we first sketch the graphs and the region in between.



The points of intersection of the two curves are,

$$\begin{aligned} x^2+2 &= 6 \\ x^2 &= 4 \end{aligned}$$

$$x = 2 \quad \text{and} \quad x = -2$$

The radius of the cylindrical shell is the distance of the shell from the axis of rotation. We draw a vertical line joining the two curves at a distance of x units from the y-axis. The cylindrical shell obtained by rotating this straight line about $x = -3$ has radius $x+3$ (the distance of the this line from $x = -3$) and the height is length of this line which is $6-(x^2+2) = 4-x^2$. Thus using the above result, we have

$$\begin{aligned} V &= 2\pi \int_0^2 (x+3)(4-x^2)\,dx \\ &= 2\pi \int_0^2 (-x^3-3x^2+4x+12)\,dx \\ &= 2\pi\Big(-\frac{x^4}{4}-x^3+2x^2+12x\Big)\Big|_0^2 = 40\pi. \end{aligned}$$

□

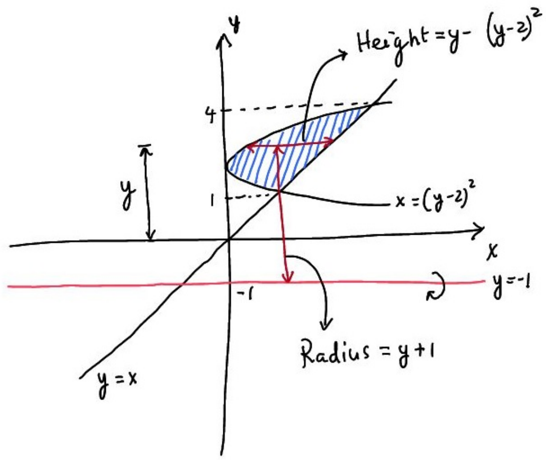
6.1.2.2. **Shell Method : Volume of revolution about a horizontal axis of a region between two curves.** When we rotate the region between two curves about a horizontal axis, the theory of cylindrical shells still remains the same. We still find the radius of the shell by finding the distance of the horizontal line joining the two curves from the axis of rotation. The height of the shell is similarly found by finding the length of this horizontal line. The only difference we have in this case is the change of variables from x to y. We integrate with respect to y, hence the equations of our original curves have to be functions of y (which is usually how it's given).

Therefore, if the region between the curves $x = f(y)$ and $x = g(y)$ with $f(y) \geq g(y)$ on $[c, d]$ is rotated about the x-axis, then the volume of rotation of the solid is

$$\boxed{V = 2\pi \int_c^d (\text{radius})(\text{height of the shell})\, dy = 2\pi \int_c^d y(f(y) - g(y))\, dy}$$

Example 6.4. Use the Shell Method to find the volume obtained by rotating the region between the curves $x = (y-2)^2$, and $y = x$ about the horizontal axis $y = -1$.

Solution. As always we first sketch the graphs and the region in between.



The points of intersection of the two curves are,

$$\begin{aligned} (y-2)^2 &= y \\ y^2 - 4y + 4 &= y \\ y^2 - 5y + 4 &= 0 \\ (y-4)(y-1) &= 0 \end{aligned}$$

$$y = 1 \quad \text{and} \quad y = 4$$

The radius of the cylindrical shell is the distance of the shell from the axis of rotation. We draw a horizontal line joining the two curves at a distance of y units from the x-axis. The cylindrical shell obtained by rotating this straight line about $y = -1$ has radius $y + 1$ (the distance of the this line from $y = -1$) and the height is length of this line which is $y - (y-2)^2 = -y^2 + 5y - 4$. Thus using the above result, we have

$$\begin{aligned} V &= 2\pi \int_1^4 (y+1)(-y^2+5y-4)\,dy \\ &= 2\pi \int_1^4 (-y^3+4y^2+y-4)\,dy \\ &= 2\pi\Big(-\frac{y^4}{4}+\frac{4y^3}{3}+\frac{y^2}{2}-4y\Big)\Big|_1^4 \\ &= \frac{63\pi}{2} \end{aligned}$$

$\square$

6.2. Work and Energy

(a) Set up an integral representing an amount of work or a hydrostatic pressure.

6.2.1. Work and Energy

The amount of work W done in moving an object with a constant force F Newtons through distance of d meters in the direction of the force is given by

$$\boxed{W = F \cdot d}$$

Now what happens if the force varies as the object moves from a to b along the x-axis. If we denote this force by $F(x)$ then as we have done countless times before, we can divide the region between a and b into N equal segments. Each segment is of length Δx and the force needed to move the object on the interval $[x_{i-1}, x_i]$ can be considered constant and equal to $F(x_i)$ as the interval is very small. Then, the work W_i that is done in moving the object from x_{i-1} to x_i is

$$W_i = F(x_i)\Delta x$$

Hence, the total work done is

$$W = \sum_{i=1}^{N} W_i \approx \sum_{i=1}^{N} F(x_i)\Delta x$$

To find the exact work we take $N \to \infty$. In this case the sum on the right converges to $\int_a^b F(x)\,dx$.

Definition 6.5. Work The work performed in moving an object along the x-axis from a to b by applying a force of magnitude $F(x)$ is

$$W = \int_a^b F(x)\,dx$$

We will be mostly concerned with the work done in stretching and compressing a spring. By **Hooke's Law**, when a spring is stretched or compressed to position x, it exerts a restoring force of magnitude kx in the opposite direction. The constant k is called the **spring constant**. We can use this fact to find the work required to stretch or compress a spring beyond its equilibrium points.

Example 6.6. If 5 J of work is needed to stretch a spring 10 cm beyond equilibrium, how much work is required to stretch it 15 cm beyond the equilibrium point.

Solution. We first find the spring constant k. We also have to make sure that the units of length are in meters. Since,

$$5 = \int_0^{0.1} kx\,dx = \left[\frac{1}{2}kx^2\right]_0^{0.1} = .005k$$

Therefore, $k = \dfrac{5}{0.005}$ N/m $=$ 1000 N/m. Since 15 cm $=$ 0.15 m, therefore the work required to stretch it is

$$W = \int_0^{0.15} kx\,dx = \int_0^{0.15} 1000x\,dx = \left[500x^2\right]_0^{0.15} = 500 \cdot 0.0225 = 11.25 \text{ J}$$

$\square$